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A Rigorous Control of Logarithmic Corrections in Four-Dimensional φ^4 Spin Systems. I. Trajectory of Effective Hamiltonians

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Using Gawędzki and Kupiainen's rigorous block spin transformation method, we study critical phenomena in φ^4 spin systems in four dimensions. In Part I of this work we investigate in detail the renormalization group trajectory of the system not exactly at the critical point.

KEY WORDS: φ^4 spin systems; rigorous block spin transformations; critical phenomena; cluster expansion.

1. INTRODUCTION

Rigorous investigations of quantum field theories and of the critical behavior in spin systems are among the most interesting and important problems in current theoretical physics. Recent developments in this field have established many fascinating features of the systems, including the absence of intermediate phase in $d>2$ dimensions,[1] the triviality of φ_d^4 field theories in $d>4$ dimensions,[2-4] the perturbative nature of those in $d<4$ dimensions,[5] and the mean-field-like behavior of critical phenomena in φ_d^4 spin systems in $d>4$ dimensions.[2,3,6] Taking all these successes into account, it seems that now one of the most important and challenging subjects in this field is to deal with the systems in marginal dimensionality $d=4$, where the triviality of φ^4 field theory and the existence of logarithmic corrections to mean-field-type critical behavior[7] are predicted. Among promising attempts in this direction,[8,9] the Kadanoff–Wilson renor-

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malization group method,[10] or its possible rigorous version, has been expected to be a powerful tool in investigating such subtle behavior of systems with infinite degrees of freedom.

Recently, in a series of papers,[11-17] Gawędzki and Kupiainen have presented a new rigorous approach to block spin transformations. Among other outstanding results, they have succeeded in tracing rigorously the renormalization group flow of the weakly coupled lattice φ_4^4 system (i.e., φ^4 spin system in four dimensions) which is *exactly* at the critical point.[16] This, in particular, elucidates the long-range behavior of critical $\lambda\varphi_4^4$ spin systems (e.g., critical exponent equality $\eta = 0$, a logarithmic correction to scaling behavior of energy–energy correlations[15]), and establishes the infrared counterpart of the triviality of $\lambda\varphi_4^4$ field theory for a sufficiently small value of λ.

In the present and following papers,[18] we extend their analysis to study φ_4^4 spin systems *not* exactly at the critical point. In particular, we study the critical phenomena that take place when the system approaches its critical point (from the high-temperature phase), and establish the existence of logarithmic corrections to the behavior predicted by the mean field theory. This is accomplished by tracing in detail the *mutual difference* between two renormalization group trajectories in the vicinity of the Gaussian fixed point.[17,19]

The present paper contains the first part of the program, and is devoted to the study of renormalization group flow in the φ_4^4 system.

In subsequent work[18] we extract logarithmic corrections from the effective potential thus obtained.

The organization of this paper is as follows. In Section 1.1 we define our model and the block spin transformation, and in Section 1.2 we list some notations. Section 2 is devoted to stating our result. In Section 2.1, we explain the basic idea of our analysis; in Section 2.2, we list the inductive assumptions used to describe the effective Hamiltonians; and in Section 2.3, we give precise statements of our results.

In Sections 3–7, we present the derivation of the result.

This paper is not self-contained; some knowledge of Gawędzki and Kupiainen's program[11,13,16] is assumed.

1.1. The Model and Block Spin Transformation

The Gibbs measure of a φ^4 spin system on a d-dimensional hypercubic lattice Λ_0 is defined as

$$d\mu(\Phi) = Z^{-1} \exp[-\mathscr{H}^0_{\Lambda_0}(\Phi)] \prod_x d\varphi_x \tag{1.1}$$

where $x \in \Lambda_0 \subset \mathbf{Z}^d$, $\varphi_x \in R$, $\mathbf{\Phi} \equiv \{\varphi_x\}_x$,

$$\mathscr{H}^0_{\Lambda_0} \equiv \frac{1}{4} \sum_{|x-y|=1} (\varphi_x - \varphi_y)^2 + \sum_x \left[\left(\frac{\mu_0}{2} - \frac{\lambda_0}{4} G^{(;\xi)}_{0xx} \right) \varphi_x^2 + \frac{\lambda_0}{4!} \varphi_x^4 \right] \quad (1.2)$$

$G^{(;\xi)}_{0xx}$ is a massless Gaussian propagator with infrared regulator ξ $(\equiv 1)$ and Z is determined by the normalization condition $\int d\mu(\mathbf{\Phi}) = 1$. For any function F of $\mathbf{\Phi}$, its *thermal expectation value* is defined by

$$\langle F \rangle \equiv \int F \, d\mu(\mathbf{\Phi}) \quad (1.3)$$

We take Λ_0 as a d-dimensional torus of side L^N.

The Kadanoff–Wilson *block spin transformation* (BST) performed in this paper is defined as the following transformation from an effective Hamiltonian $\mathscr{H}(\mathbf{\Phi})$ to another $\mathscr{H}'(\mathbf{\Phi})$:

$$e^{-\mathscr{H}'(\Phi)} \equiv \mathscr{N} \int \prod_y d\varphi_y \prod_{x_1 \in \mathbf{Z}^d} \delta\left(\varphi'_{x_1} - \zeta^{-1/2} L^{-(d+2)/2} \sum_{y \in B(x_1)} \varphi_y \right) \cdot e^{-\mathscr{H}(\Phi)} \quad (1.4)$$

where φ' is a block spin variable, $x_1 \in \Lambda_1$, and $B(x_1)$ is an L^d-block $(\subset \Lambda_0)$ centered at Lx_1. Here $\mathscr{N}$ is a suitable normalization factor (so that $e^{-\mathscr{H}'(0)} = 1$), and ζ is the ratio of wave function renormalization constants

$$\varphi^n_x \equiv z_n^{-1/2} L^{-3n} \sum_{y \in B^n(x)} \varphi_y \equiv z_n^{-1/2} (\hat{c}^n \varphi)_x$$

$$\zeta_n \equiv z_{n+1}/z_n$$

We usually denote the first effective Hamiltonian by $\mathscr{H}^0$ and the nth one by $\mathscr{H}^n$, but sometimes abbreviate $\mathscr{H}^n$ as $\mathscr{H}$, and $\mathscr{H}^{n+1}$ as $\mathscr{H}'$. Our aim is to investigate $\mathscr{H}^n$ in detail. In this paper, as in Ref. 16, we perform estimates that are *uniform* in the volume of the original lattice Λ_0. So in the following analysis, we assume Λ_0 to be sufficiently large so that the block spin transformation considered can be performed. (E.g., if we are to perform BST n times, we take $N \geqslant N_0 + n$.)

1.2. Notations

We use mostly the same notations as in Ref. 16. We denote field configuration $\{\varphi_x\}_{x \in X}$ (X is a subset of $\mathbf{Z}^d$) by $\mathbf{\Phi}$.

Δ_0: four-dimensional torus $(\subset \mathbf{Z}^4)$ with side L^N, $N > 0$

Λ_n: four-dimensional torus $(\subset \mathbf{Z}^4)$ with side L^{N-n}

We also denote the *number of sites* in Λ_n by Λ_n.

Δ: four-dimensional hypercube of side L^{N_0}, with $N_0 = 8$

$|\Lambda_n|$: the *number of Δ's* in Λ_n $(= L^{4(N-N_0-n)})$

$|x|$, $x \in \mathbf{Z}^4$: the metric on $\mathbf{Z}^4$, defined as $|x| \equiv \sum_{\mu=1}^{4} |x_\mu|$, for $x \in \mathbf{Z}^4$ (our definition is the same as in Ref. 13, but is twice that of Ref. 16)

$|x|_\infty$, $x \in Z^4$: $|x|_\infty \equiv \max_\mu |x_\mu|$

$|X|$, X is a paved set: the number of L^{4N_0} blocks Δ in X.

$\mathscr{L}(X)$, X is a paved set: the length of the shortest tree connecting the centers of Δ's building X

$\mathscr{H}^n$: the effective Hamiltonian after n BST

$d\mu_G$: normalized Gaussian measure with mean zero, covariance G

$G_n^{(;L^{2n}\xi)}$: massless Gaussian propagator with infrared regulator $L^{2n}\xi$ after n BSTs applied to $G_0 \equiv (-\Delta)^{-1}$ [see (2.13) of Ref. 16 and Appendix. Also see note added in proof.]

$\mathscr{A}_n$, Q_n, $\mathscr{T}_n$: the same as in Ref. 16

We also use massive versions of these Gaussian propagators and kernels.

$G_n^{(\mu_n)}$: massive Gaussian propagator obtained by applying BST n times to

$$G_0^{(L^{-2n}\mu_0)} \equiv (\Delta + L^{-2n}\mu_n)^{-1}$$

Note that μ_n refers to the expected mass on Λ_n, not on Λ_0.

$\mathscr{A}^{(\mu_n)}$, $Q^{(\mu_n)}$, and $\mathscr{T}^{(\mu_n)}$ are defined in the same way.

$$I_n^{(\mu_n)} \equiv \int_{\square_0} dx \int dy\, (L^2 \mathscr{T}_n^{(\mu_n)})_{Lx\,Ly} (L^2 \mathscr{T}_n^{(\mu_n)} + 2L^2 Q_n^{(\mu_n)})_{Lx\,Ly} \tag{1.5}$$

$I_\pm$: Bounds on I_n (and $I_n^{(\mu_n)}$, $\mu_n \geq 0$); $I_- \leq I_n \leq I_+$ (see Section 2.1)

Quantities without the superscript (μ_n) are *massless* ones.

As for field configurations, we use:

$\mathscr{K}_n(X)$: same as (4.1) of Ref. 16

$D_n(\mathbf{\Psi})$: same as (4.10) of Ref. 16

$$\mathscr{D}_{n,m}^{(\mu_n)}(D, X) \equiv \{\mathscr{A}_n^{(\mu_n)} \mathbf{\Phi}^n|_X + \mathbf{\Psi}^n : D_m(\mathscr{A}_n^{(\mu_n)} \mathbf{\Phi}^n) \subset D, \mathbf{\Psi}^n \subset \mathscr{K}_m(X)\}$$

We abbreviate $\mathscr{D}_{n,n}^{(\mu_n)}$ as $\mathscr{D}_n^{(\mu_n)}$.

We sometimes omit the subscript or superscript n when it is obvious from the context.

2. MAIN RESULTS

In this section we state our main results concerning the noncritical trajectory. Because the statements themselves are rather complicated, we first (Section 2.1) describe our general idea of treating the noncritical trajectory, then state inductive assumptions used to describe the trajectory, and finally state the theorems in terms of these inductive assumptions.

2.1. General Framework of the Analysis

There are two problems in treating the noncritical trajectory.

The first one is typical of treatments of a critical (or nearly critical) theory: We do not know the exact value of the critical point μ_c.

The second one is specific to the noncritical trajectory: The mass term is *relevant*, and will grow like L^{2n}!!

The first problem is solved by analyzing a noncritical trajectory in terms of the critical one, rather than analyzing the noncritical one directly. That is, we investigate the *mutual difference* between critical and noncritical trajectories very carefully.(17,19) Then, since the result of Gawędzki and Kupiainen gives us sufficient information on the critical trajectory, we can obtain sufficient information on the noncritical one as well.

The second problem is solved by performing perturbation expansion around massive Gaussians. (Recall that the result of Gawędzki and Kupiainen showed that critical theory = massless Gaussian + small correction.) That is, at each step of the iteration, we perform the "mass renormalization" (as well as the "wave function renormalization") to eliminate the mass term, and express the noncritical theory as massive Gaussian + small corrections, *without mass term.* It is the $(\text{mass})^2$ of the Gaussian propagator that grows like L^{2n}, and we can perform the perturbation expansion for this "small correction."

We perform the actual analysis by dividing n (number of iterations) into three regions (see Fig. 1):

Region I: The "mass term" is quite small. Here we take the mutual difference between critical and noncritical trajectory, expressing both as massless Gaussian + small corrections.

Region II: The "mass" is not so small nor so large. Here we perform the perturbation around massive Gaussians by expressing the effective Hamiltonian as massive Gaussian + small correction.

Region III: The mass is extraordinarily large. Here we do almost the same thing as in region II, but we have to take the "large mass" into account carefuly.

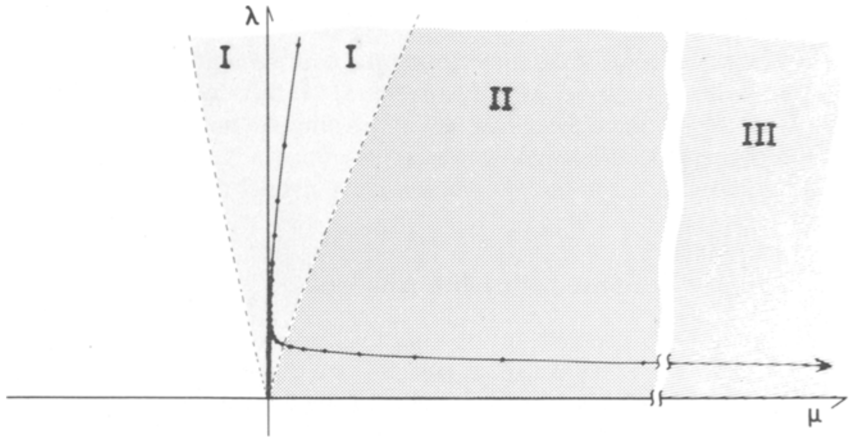

Fig. 1. A schematic view of a renormalization group trajectory and the regions I, II, and III.

Throughout the following analyis, we fix the initial value of λ to a constant, and vary only the initial value of μ. The initial value λ_0 is chosen sufficiently small so that the proof of Theorems 2.1 and 2.2 can be carried out (so that the bound $\mathrm{B1}_0$ holds for λ_0 for n_0 of Section 2.2).

2.2. Inductive Assumptions and Bounds

Here we list inductive assumptions and bounds used to describe the effective Hamiltonians. There inductive assumptions and bounds are rather complicated, so the bored reader can first skip to Section 2.3.

First we list our choice of various constants. The explicit values should not be taken seriously, because they are far from optimal.

Choice of Constants. $0<\alpha\equiv\alpha(L)$: sufficiently small: $\beta\equiv 6\alpha$; $L\geqslant\bar{L}\equiv 513$: odd integer; $I_-\equiv 10^{-2}$, $I_+\equiv\frac{1}{3}\ln L+37/2$; $C_+\equiv I_-^{-1}$; $C_-\equiv(2I_+)^{-1}$; $N_0\geqslant\bar{N}_0(L)$: large; $C_0\geqslant\bar{C}_0(L)$; $C_2\geqslant\bar{C}_2(L, N_0)$; $C_1\geqslant\bar{C}_1(L, N_0, C_2)$; and $\bar{n}_0\geqslant\bar{n}_0(L, N_0, C_0, C_1, C_2)$, with $\bar{C}_0$, $\bar{C}_1$, $\bar{C}_2$, and $\bar{n}_0$ sufficiently large as in Ref. 16, e.g.,

$$\bar{n}_0\approx\exp(L^5) \tag{2.1}$$

Now the effective Hamiltonians of critical theory and of noncritical theory (in region I) are expressed as follows.

A_n. General Properties of the Effective Hamiltonian

$$\exp[-\mathscr{H}^n(\Phi^n)] = \exp[-\tfrac{1}{2}(\Phi^n, G_n^{-1}\Phi^n)] \exp[-V^n(\Psi^n)]|_{\Psi^n = \mathscr{A}_n\Phi^n}$$
$$\exp[-V^n(\Psi^n)] = \exp[-V_2^n(\Psi^n)] \exp[-V_{\geqslant 4}^n(\Psi^n)]$$

$V_2^n(\Psi)$ and $\exp[-V_{\geqslant 4}^n(\Psi^n)]$ are analytic in Ψ on $\mathscr{D}_n(D, L^{-n}\Lambda)$.

A1

$$V_2^n(\Psi^n) = \int dx \left(\frac{\mu_n}{2} - \frac{\lambda_n}{4}\right) \mathscr{G}_{nxx}^{(;L^{2n}\xi)}(\psi_x^n)^2$$
$$+ \sum_{\mu,\nu} \int dx \int dy \, (\mathscr{K}_n^{\mu\nu})_{xy}(\partial_\mu \psi_x^n - \partial_\mu \psi_y^n)(\partial_\nu \psi_y^n)$$

A2. The term $\exp(-V_{\geqslant 4}^n)$ is analytic on $\mathscr{D}_n(D, L^{-n}\Lambda)$ and has the representation

$$\exp[-V_{\geqslant 4}^n(\Psi^n)] = \sum_{\{X_i\}} \left[\prod_i g_{X_i}^{n,D}(\Psi^n)\right] \exp[-V_{\sim D}^{\{X_i\}}(\Psi^n)]$$

Here $\sum_{\{X_i\}}$ runs over the collections of paved sets such that $X_i \cap X_j = \varnothing$ $(i \neq j)$, $D \subset (\bigcup X_i)$, each $D \cap X_i$ is a nonempty union of connected components of D.

A2a. The term $g_X^{n,D}(\Psi^n)$ depends only on $\Psi^n|_X$, even in ψ^n, analytic on $\mathscr{D}_n(D, X)$, and satisfies the bounds to be specific later.

A2b. The term $V_{\sim D}^{\{X_i\}}$ depends only on $\Psi^n|_{\sim D}$, is analytic on $3\mathscr{K}_n(L^{-n}\Lambda \backslash D)$, and has a representation

$$V_{\sim D}^{\{X_i\}}(\Psi^n) = \frac{\lambda_n}{4!} \int_{\sim D} dx \, (\psi_x^n)^4 + \sum_{Y \subset \sim \bigcup X_i} \tilde{V}_{4Y}(\Psi^n) + \sum_{Y \subset \sim \bigcup X_i} V_{\geqslant 6Y}(\Psi^n)$$

where $\tilde{V}_{4Y}(\Psi^n)$ is the restriction to the diagonal of a quartic nonsymmetric form $\tilde{V}_{4Y}(\Psi_1^n, \Psi_2^n, \Psi_3^n, \Psi_4^n)$. Here $\tilde{V}_{4Y}(\Psi_1^n, \Psi_2^n, \Psi_3^n, \Psi_4^n)$ depends only on $\Psi|_Y$, and Ψ_4^n enters $\tilde{V}_{4Y}(\Psi_1^n, \Psi_2^n, \Psi_3^n, \Psi_4^n)$ only through its differences at each pair of points, and

$$V_{\geqslant 6Y}(\Psi^n) = -\frac{\lambda_n^2}{72} \sum_{(\Delta_1, \Delta_2)} \int_{\Delta_1} dx \int_{\Delta_2} dy \, (Q_n)_{xy}(\psi_x^n)^3(\psi_y^n)^3 + \tilde{V}_{\geqslant 6Y}(\Psi^n)$$

Here, $\tilde{V}_{\geqslant 6Y}(\Psi^n)$ depends only on $\Psi|_Y$, is analytic on $3\mathscr{K}_n(Y)$, and has the Taylor series starting with sixth term. As in Ref. 16, $\Sigma_{(\Delta_1, \Delta_2)}$ means that the term appears only when $\Delta_1 \cup \Delta_2 = Y$. (Otherwise, it should be regarded as zero.)

Inductive bounds to be used with the above assumptions are as follows.

B_n. Inductive Bounds (on Critical Theories)

B1

$$\frac{C_-}{n_0+n} \leqslant \lambda_n \leqslant \frac{C_+}{n_0+n}$$

B2

$$|\mu_n| \leqslant (n_0+n)^{-3/2}$$

$$\int_{\Delta_1} dx \int_{\Delta_2} dy \, |\mathscr{K}^{\mu\nu}_{n,xy}| \cdot |x-y|^{2/3} \leqslant (n_0+n)^{-3/2} \exp[-\alpha \mathscr{L}(\Delta_1 \cup \Delta_2)]$$

B3. On $\mathscr{D}_n(D, X)$,

$$|g^{n,D}_X(\Psi^n) \leqslant \exp\left[C_2 |D \cap X| - (\lambda_n/24)^{1/2} \int_{D \cap X} dx \, |\psi^n_x|^2 + \lambda_n \int_{D \cap X} dx \, (\operatorname{Im} \psi^n_x)^4 - \alpha \mathscr{L}(x) \right]$$

B4. On $3\mathscr{K}_n(Y)$,

$$|\tilde{V}_{4Y}(\Psi^n)| \leqslant (n_0+n)^{-3/4} \exp[-\alpha \mathscr{L}(Y)]$$
$$|\tilde{V}_{\geqslant 6Y}(\Psi^n)| \leqslant (n_0+n)^{-2/3} \exp[-\alpha \mathscr{L}(Y)]$$

B5

$$|\zeta_{n-1} - 1| \leqslant (n_0+n)^{-3/2}$$

C_n. Inductive Bounds (on the Mutual Difference in Region I)

Here mutual differences of various quantities of critical and noncritical theories are bounded. λ_n stands for $(\lambda_n)_{\text{critical}}$.

C1

$$|(\lambda_n)^{nc} - (\lambda_n)_c| \leqslant (n_0+n)^{-1/3} \lambda_n |\Delta \mu_n|$$

$$\int_{\Delta_1} dx \int_{\Delta_2} dy \, |x-y|^{2/3} |(\mathscr{K}^{\mu\nu}_{n,xy})_{nc} - (\mathscr{K}^{\mu\nu}_{n,xy})_c| \leqslant |\Delta \mu_n| \, (n_0+n)^{-2/3} \exp[-\alpha \mathscr{L}(\Delta_1 \cup \Delta_2)]$$

where $\Delta \mu_n \equiv (\mu_n)_{nc} - (\mu_n)_c$.

C2. On $\mathscr{D}(D, X)$,

$$|(g_X^{n,D})_{nc} - (g_X^{n,D})_c| \leqslant |\Delta\mu_n| (n_0 + n)^{1/2} \exp\left[C_2 |D \cap X| - (\lambda_n/24)^{1/2} \int_{D \cap X} |\psi_x^n|^2 + \lambda_n \int_{D \cap X} dx\, (\operatorname{Im} \psi_x^n)^4 - \alpha \mathscr{L}(x)\right]$$

C3. The difference of $\tilde{V}_{4Y}$ can be written as

$$\tilde{V}_{4Y}(\boldsymbol{\Psi}^n)_{nc} - \tilde{V}_{4Y}(\boldsymbol{\Psi}^n)_c$$

$$= \Delta \tilde{\tilde{V}}_{4Y}(\boldsymbol{\Psi}^n) - \Delta\mu_n \frac{\lambda_n}{6} \sum_{(\Delta_1, \Delta_2)} \int_{\Delta_1} dx \int_{\Delta_2} dy\, (Q_n)_{xy} (\psi_x^n - \psi_y^n)(\psi_y^n)^3$$

with bounds on $\Delta \tilde{\tilde{V}}_{4Y}$ and $\Delta V_{\geqslant 6Y}$ on $3\mathscr{K}_n(Y)$:

$$|\Delta \tilde{\tilde{V}}_{4Y}(\boldsymbol{\Psi}^n)| \leqslant |\Delta\mu_n| (n_0 + n)^{-1/3} \exp[-\alpha L(Y)]$$

$$|\Delta V_{\geqslant 6Y}(\boldsymbol{\Psi}^n)| \leqslant |\Delta\mu_n| \exp[-\alpha \mathscr{L}(Y)]$$

C4

$$|(\zeta_{n-1})_{nc} - (\zeta_{n-1})_c| \leqslant (n_0 + n)^{-7/9} |\Delta\mu_n|$$

The second set of assumptions and bounds describes noncritical theories not close to the critical one.

D_n. *General Properties of Effective Hamiltonians (in Region II)*
We have

$$\exp[-\mathscr{H}^n(\boldsymbol{\Phi}^n)] = \exp[-\tfrac{1}{2}(\boldsymbol{\Phi}^n, G_n^{(\tilde{\mu}_n)-1} \boldsymbol{\Phi}^n)] \exp[-V^n(\boldsymbol{\Psi}^n)]|_{\boldsymbol{\Psi}^n = \mathscr{A}_n^{(\tilde{\mu}_n)} \boldsymbol{\Phi}^n}$$

and $\exp[-V^n(\boldsymbol{\Psi}^n)]$ is analytic on $\mathscr{D}_n^{(\tilde{\mu}_n)}(D, L^{-n}\Lambda)$ and can be written as

$$\exp[-V^n(\boldsymbol{\Psi}^n)] = \exp[-V_2^n(\boldsymbol{\Psi}^n)] \exp[-V_{\geqslant 4}^n(\boldsymbol{\Psi}^n)]$$

D1

$$V_2^n(\boldsymbol{\Psi}^n) = -\int dx \frac{\lambda_n}{4} \mathscr{G}_{nxx}^{(\mu_n; L^{2n}\xi)} (\psi_x^n)^2 + \sum_{\mu, \nu} dx \int dy\, (\mathscr{K}_n^{\mu\nu})_{xy} (\partial_\mu \psi_x^n - \partial_\mu \psi_y^n)(\partial_\nu \psi_y^n)$$

D2. Same as A2, except that we use $\mathscr{D}_n^{(\tilde{\mu}_n)}$, $\mathscr{A}_n^{(\tilde{\mu}_n)}$, and $Q_n^{(\tilde{\mu}_n)}$ instead of $\mathscr{D}_n$, $\mathscr{A}_n$, and Q_n.

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E_n. *Inductive Bounds (in Region II)*

E1

$$\frac{C_-/2}{n_0+n} \leqslant \lambda_n \leqslant \frac{2C_+}{n_0+n}, \qquad \tilde{\mu}_n \geqslant (n_0+n)^{-1}$$

E2

$$\int_{\Delta_1} dx \int_{\Delta_2} dy\, |\mathscr{K}^{\mu\nu}_{n,xy}| \cdot (x-y)^{2/3} \leqslant (n_0+n)^{-3/2} \exp[-\alpha \mathscr{L}(\Delta_1 \cup \Delta_2)]$$

E3. On $\mathscr{D}_n^{(\tilde{\mu}_n)}(D, L^{-n}\Lambda)$,

$$|g_X^{n,D}(\Psi^n)| \leqslant \exp\left[C_2 |D \cap X| - \left(\frac{\lambda_n}{24}\right)^{1/2} \int_{D \cap X} dx\, |\psi_x^n|^2 + \lambda_n \int_{D \cap X} dx\, (\operatorname{Im} \psi_x^n)^4 - \alpha \mathscr{L}(X) \right]$$

E4. On $3\mathscr{K}_n(Y)$,

$$|\tilde{V}_{4Y}(\Psi^n)| \leqslant (n_0+n)^{-3/4} \exp[-\alpha \mathscr{L}(Y)]$$
$$|\tilde{V}_{\geqslant 6Y}(\Psi^n)| \leqslant (n_0+n)^{-2/3} \exp[-\alpha \mathscr{L}(Y)]$$

E5

$$|\zeta_{n-1} - 1| \leqslant (n_0+n)^{-3/2}$$

F_n. *General Properties of Effective Hamiltonians (in Region III)*

Same as D_n, except that we use $\mathscr{D}_{n,n_1'}^{(\tilde{\mu}_n)}$, $\mathscr{K}_{n_1'}$ instead of $\mathscr{D}_n^{(\tilde{\mu}_n)}$, $\mathscr{K}_n$.

G_n. *Inductive Bounds (in Region III)*

Same as E_n, except that we use n_0+n_1', $\mathscr{D}_{n,n_1'}^{(\tilde{\mu}_n)}$, $\mathscr{K}_{n_1'}$ instead of n_0+n, $\mathscr{D}_n^{(\tilde{\mu}_n)}$, $\mathscr{K}_n$. That is

G1

$$\frac{C_-/3}{n_0+n_1'} \leqslant \lambda_n \leqslant \frac{3C_+}{n_0+n_1'}, \qquad \tilde{\mu}_n \geqslant \tilde{\mu}_{n_1'} \left(\frac{L^2}{2}\right)^{n-n_1'}$$

G2

$$\int_{\Delta_1} dx \int_{\Delta_2} dy\, |\mathscr{K}^{\mu\nu}_{n,xy}| \cdot |x-y|^{2/3} \leqslant (n_0+n_1')^{-3/2} \exp[-\alpha \mathscr{L}(\Delta_1 \cup \Delta_2)]$$

G3. On $\mathscr{D}^{(\tilde{\mu}_n)}_{n,n'_1}(D, L^{-n}\Lambda)$,

$$|g^{n,D}_X(\mathbf{\Psi}^n)| \leqslant \exp\left[C_2|D\cap X| - \left(\frac{\lambda_n}{24}\right)^{1/2}\int_{D\cap X} dx\,|\psi^n_x|^2 + \lambda_n\int_{D\cap X} dx\,(\operatorname{Im}\psi^n_x)^4 - \alpha\mathscr{L}(X)\right]$$

G4. On $3\mathscr{K}_{n'_1}(Y)$,

$$|\tilde{V}_{4Y}(\mathbf{\Psi}^n)| \leqslant (n_0+n'_1)^{-3/4}\exp[-\alpha\mathscr{L}(Y)]$$
$$|\tilde{V}_{\geqslant 6Y}(\mathbf{\Psi}^n)| \leqslant (n_0+n'_1)^{-2/3}\exp[-\alpha\mathscr{L}(Y)]$$

G5

$$|\zeta_{n-1}-1| \leqslant (n_0+n'_1)^{-3/2}(2/L)^{(n-n'_1)/2}$$

R. Recursion Relations

We also list the recursion relations of λ_n, μ_n, etc.

R1

$$\lambda_{n+1}-\lambda_n = \lambda_n^2(-\tfrac{3}{2}I_n+\delta\lambda^1), \qquad |\delta\lambda^1| \leqslant (n_0+n)^{-1/9}$$

R2

$$\mu_{n+1} = L^2\mu_n+\delta\mu^1, \qquad |\delta\mu^1| \leqslant (n_0+n)^{-13/8}$$

RD. Recursion for Mutual Difference (Region I)

$$\Delta\mu_{n+1} = \Delta\mu_n L^2(1-\tfrac{1}{2}\lambda_n I_n+\tfrac{1}{2}\delta\mu^2)$$
$$|\delta\mu^2| \leqslant (n_0+n)^{-10/9}+(n_0+n)^{-1/6}|\Delta\mu|^{2/3}$$

Here λ_n stands for $(\lambda_n)_c$.

RM. Recursion for Noncritical Theory (Region II)

$$\tilde{\mu}_{n+1} = L^2\tilde{\mu}_n+\delta\mu^3, \qquad |\delta\mu^3| \leqslant (n_0+n)^{-13/8}$$
$$\lambda_{n+1}-\lambda_n = \lambda_n^2(-\tfrac{3}{2}I_n^{(\tilde{\mu}_n)}+\delta\lambda^3), \qquad |\delta\lambda^3| \leqslant (n_0+n)^{-1/9}$$

RN. Recursion for Noncritical Theory (Region III)

$$\tilde{\mu}_{n+1} = L^2\tilde{\mu}_n+\delta\mu^5, \qquad |\delta\mu^5| \leqslant (n_0+n)^{-13/8}(2/L)^{(n-n'_1)}$$
$$\lambda_{n+1} = \lambda_n+\delta\lambda^5, \qquad |\delta\lambda^5| \leqslant (n_0+n)^{-15/8}(2/L)^{(n-n'_1)}$$

2.3. Main Results

Now we can state our main theorems. As was stated at the end of Section 2.1, we fix the initial value of λ to a constant throughout the following analysis, and vary only the initial values of μ. The initial value of λ is chosen sufficiently small that the proof of Theorem 2.1 and Proposition 3.1 can be carried out. (Of course there exists such $\lambda > 0$. See Section 3.)

First let us recall the result of Gawędzki and Kupiainen, which characterizes the behavior of critical theories:

Theorem 2.1 (Gawędzki and Kupiainen[16]). Consider a theory on Λ_0, and fix the initial value of λ sufficiently small. Then there exists (at least one) *critical value* $\mu_c(\lambda; \Lambda_0)$, and if we start BST from the initial value (μ_c, λ) then $e^{-\mathscr{H}^n}$ $(n \leqslant N - N_0,\ \Lambda_0 \equiv L^{4N})$ satisfies the above assumptions A_n together with the above bounds B_n.

Remark. For finite Λ_0, $\mu_c(\lambda; \Lambda_0)$ is not unique, but exists in some domain of **R**. For the infinite-volume limit, see Theorem 2.5.

For noncritical theories, which are of particular concern in this paper, the following is proven.

Theorem 2.2 (Trajectory of noncritical theories). Consider a theory on Λ_0, fix λ sufficiently small as in Theorem 2.1, and consider a noncritical trajectory of $\mathscr{H}^n$ starting from (μ, λ). Denote $t = \mu - \mu_c(\lambda, \Lambda_0)$. Then for $n \leqslant N - N_0$ we have the following results.

Case a. $0 < t \leqslant (n_0)^{-1}$.
There exists $n_1 \geqslant 1$ specified by (2.2).

(i) Region I. For $n \leqslant n_1$, $e^{-\mathscr{H}^n}$ satisfies the assumptions A_n, and the difference between this noncritical theory and the critical one satisfies the bounds C_n. Here n_1 is defined as the minimum positive integer such that

$$\Delta\mu_{n_1 - 1} \leqslant (n_0 + n - 1)^{-1}, \qquad \Delta\mu_{n_1} > (n_0 + n_1)^{-1} \tag{2.2}$$

Moreover, for $0 \leqslant k$, $n + k \leqslant n_1$,

$$(L^2/2)^k \leqslant \Delta\mu_{n+k}/\Delta\mu_n \leqslant (2L^2)^k \tag{2.3}$$

(ii) Region II. For $n_1 + 1 \leqslant n \leqslant n_1 + \frac{9}{10}(n_0 + n_1)$ $(\equiv n'_1)$, $e^{-\mathscr{H}^n}$ satisfies the assumptions D_n together with the bounds E_n.

(iii) Region III. For $n'_1 \leqslant n$, $\mathrm{e}^{-\mathscr{H}^n}$ satisfies the assumptions F_n together with the bounds G_n.

Case b. $t > (n_0)^{-1}$.
In this case, the trajectory starts from (ii) or (iii).

Case c. $-(n_0)^{-1} \leqslant t < 0$.
As long as $n \leqslant n_1$, the trajectory is specified by (i).

The proof is presented in Sections 3–5.

Remarks.

1. We present a schematic view to the trajectory in Fig. 2. Note that $\mathscr{H}^n$ stays in the vicinity of the Gaussian fixed point (i.e., in region I) for arbitrarily long time as $t \to \pm 0$. (See also Theorem 2.3.) In other words, the behavior of $\mathscr{H}^n$ in region I will play an essential role in the study of critical phenomena (even in the low-temperature phase!).

2. If one is interested only in the trajectory with $t > 0$, it is possible to analyze it in a somewhat simpler way. That is, one can combine the method used in region I and that used in region II, and analyze the trajectory in regions I and II in a unified manner. But such a method cannot be used to analyze the trajectory for $t < 0$. For this reason, we perform the analysis in three steps.

3. The trajectory of a hierarchical model in region III was investigated by Ito.[20]

4. The BST in a uniform magnetic field is exactly the same as that without one, except that the magnetic field itself grows like L^{3n}. (How to extract physical information is a distinct problem.[21])

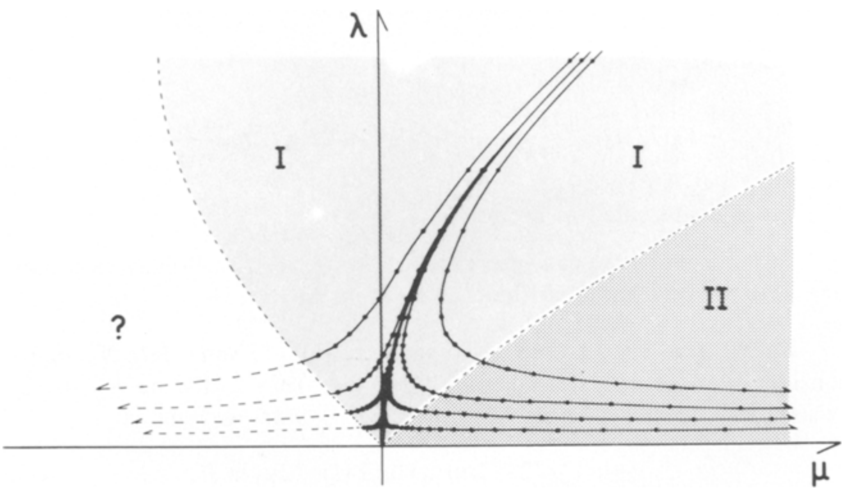

Fig. 2. A magnified view of the renormalization group trajectories around the Gaussian fixed point. The dots indicate $\mathscr{H}^n$ at every 10^8, say, iterations. It can be seen that $\mathscr{H}^n$ stays in the vicinity of the Gaussian fixed point (in region I) for arbitrary long time as $t \to \pm 0$.

5. Investigation of the trajectory in the low-temperature phase (outside region I) is quite an interesting open problem.

We can also prove more refined bounds on the behavior of μ_n and λ_n.

Theorem 2.3 (Refined Bounds on Trajectory):

(i) When $|t| < (n_0)^{-1}$, n_1 [defined in Theorem 2.2 and Eq. (2.2)] satisfies

$$\frac{|\ln|t|| - \ln n_0}{2\ln L + 1} \leqslant n_1 \leqslant \frac{|\ln|t|| - \ln n_0}{2\ln L - 1} + 1 \tag{2.4}$$

(ii) When $|t| \leqslant n_0^{-1} L^{70} e^{35}$ and $33 \leqslant n \leqslant n_1 + 1$, $\Delta\mu_n$ satisfies

$$tL^{2n} n^{-1/3} R_1(L, N_0, n_0)^{-1} \leqslant \Delta\mu_n \leqslant tL^{2n} n^{-1/3} R_1(L, N_0, n_0) \tag{2.5}$$

where

$$R_1(L, N_0, n_0) \equiv C(L, N_0)(n_0)^{C'(L, N_0)}$$

with C and C' finite positive constants depending only on L and N_0.

(iii) When $0 < t < (n_0)^{-1}$, $\tilde{\mu}_n$ of assumptions D_n satisfies, for

$$0 \leqslant m \leqslant \frac{9}{10}\left(n_0 + \frac{|\ln|t|| - \ln n_0}{2\ln L + 1}\right)$$

the condition

$$\frac{1}{2} R_1(L, n_0)^{-1} \frac{tL^{2n_1+2m}}{(n_1)^{1/3}} \leqslant \tilde{\mu}_{n_1+m} \leqslant 2R_1(L, n_0) \frac{tL^{2n_1+2m}}{(n_1)^{1/3}} \tag{2.6}$$

The proof is presented in Section 6.

As a corollary, we can prove the following, which will play an essential role in extracting the logarithmic corrections in Part II.

Corollary 2.4 (To Be Used in Part II). Given $M(L, N_0, n_0) > 1$, consider the noncritical trajectory for $0 < t < (n_0)^{-1}$. Define $n_2 \geqslant 0$ (if it exists) as the smallest integer such that $\tilde{\mu}_{n_1+n_2} \geqslant M$. As long as

$$-\ln t \geqslant (2 + 1/\ln L)\ln(2M) - 2n_0 \ln L$$

then n_2 exists, and

$$M \leqslant \tilde{\mu}_{n_1+n_2} \leqslant 2L^2 M \tag{2.7}$$

Moreover, $n_1 + n_2$ and M are related by

$$\frac{M}{2R_1} \leqslant \frac{tL^{2n_1+2n_2}}{(n_1)^{1/3}} \leqslant 2L^2 M R_1 \tag{2.8}$$

So far, we have been considering a theory and BST on Λ_0. Let us now consider the infinite-volume limit ($\Lambda_0 \to \mathbf{Z}^4$) of $\mu_c(\lambda; \Lambda_0)$. We have

Theorem 2.5. Consider a sequence of $\mu_c(\lambda; \Lambda_0)$ of Theorem 2.1 for fixed λ. As we let $\Lambda_0 \to \mathbf{Z}^4$, (the existing domain of) $\mu_c(\lambda; \Lambda_0)$ *converges to a point in* $\mathbf{R}$. We denote this point by $\mu_c(\lambda)$:

$$\lim_{\Lambda_0 \to \mathbf{Z}^4} \mu_c(\lambda; \Lambda_0) = \mu_c(\lambda) \tag{2.9}$$

The proof is presented in Section 7.

Remark. As mentioned, for finite Λ_0, $\mu_c(\Lambda_0)$ is not unique, but exists in some range of $\mathbf{R}$. The above proposition tells us that the existing domain of $\mu_c(\Lambda_0)$ shrinks to a unique point of $\mathbf{R}$ as $\Lambda_0 \to \mathbf{Z}^4$.

3. TRAJECTORY IN REGION I

Here we fix the initial value of λ suficiently small (as in Theorems 2.1 and 2.2) and vary only μ. We consider the case $|t| \leqslant (n_0)^{-1}$.

Theorem 2.2(i) is proven by the following inductive proposition.

Proposition 3.1. Suppose $(\mathscr{H})_c$ satisfies A_n and B_n, $(\mathscr{H})_{nc}$ satisfies A_n, and the difference satisfies C_n. Then as long as $|\Delta\mu| \leqslant (n_0 + n)^{-1}$, $(\mathscr{H}')_{nc}$ satisfies A_{n+1} and C_{n+1} together with the recursion $\mathrm{R1}_{n \to n+1}$ and $\mathrm{RD}_{n \to n+1}$.

Proof of Theorem 2.2(i) Assuming Proposition 3.1. Because Proposition 3.1 itself iterates, we have only to show the existence of n_1 and the rough bound (2.3). But these can be easily derived from RD as follows: As $|\lambda_n I_n - \delta\mu^2|$ is extremely small, RD immediately yields

$$L^2/2 \leqslant \Delta\mu_{n+1}/\Delta\mu_n \leqslant 2L^2$$

This means that ($L \geqslant 500$) $|\Delta\mu|$ is monotone increasing in n, while $(n_0 + n)^{-1}$ is monotone decreasing. Thus, n_1, specified by (2.2), exists and the bound (2.3) holds. ∎

3.1. Proof of Proposition 3.1

Because Proposition 3.1 is a natural extension of the result of Gawędzki and Kupiainen, we only present a rough sketch here. The $(n+1)$th effective Hamiltonian is given by [Ref. 16, (2.38), (2.39); also see note added in proof].

$$\exp[-\mathscr{H}'(\Phi')] = \exp[-\tfrac{1}{2}(\Phi', G_{n+1}^{-1}\Phi')] \exp[-V'(\Psi')]|_{\Psi'=\mathscr{A}'\Phi'}$$

$$\exp[-V'(\Phi')] = \exp\left[\tfrac{1}{2}(1-\zeta)\int dx\,(\partial_\mu \psi'_x)^2\right]$$
$$\times \int d\mu_1(Z)\,\{\exp[-V(L^{-1}\zeta^{1/2}\Psi' + \mathscr{Z})]\}/(\Psi'=0) \quad (3.1)$$

Now the basic idea is quite simple: Take the difference of *everything*! That is, we write down expressions for various quantities for both critical and noncritical theories after Ref. 16, and estimate their differences. How to proceed will be almost clear from the form of inductive bounds C.

For example, to take the difference of $(W'_\Delta)_{nc}$ and $(W'_\Delta)_c$, we write each of them as [compare Eq. (7.4) of Ref. 16]

$$\begin{aligned}
&(W'_\Delta)_{nc} - (W'_\Delta)_c \\
&\quad = \langle (V_{L\Delta})_{nc} - (V_{L\Delta})_c \rangle_0 \\
&\qquad - \tfrac{1}{2}\{\langle (V_{L\Delta})_{nc}; (V_{L\Delta})_{nc}\rangle_0 - \langle (V_{L\Delta})_c; (L_{L\Delta})_c\rangle_0 \\
&\qquad + \tfrac{1}{2}\int ds\,(1-s)^2 \{\langle (V_{L\Delta})_{nc}; (V_{L\Delta})_{nc}; (V_{L\Delta})_{nc}\rangle_{s,nc} \\
&\qquad - \langle (V_{L\Delta})_c; (V_{L\Delta})_c; (V_{L\Delta})_c\rangle_{s,c}\} \\
&\qquad + O_0\{\exp[-\varepsilon(n_0+n)^{1/2}]\}_{nc} - O_0\{\exp[-\varepsilon(n_0+n)^{1/2}]\}_c
\end{aligned}$$

and estimate each term using the bounds B for critical theories and C for differences.

Other quantities are treated in a similar way, using various interpolation formulas to connect the critical theory $(s=0)$ to the noncritical one $(s=1)$. We mate two remarks.

Remark 1 (Concerning the critical trajectory). The method of Ref. 16 does not yield the desired bound on $\tilde{V}_6$. We have to treat $\tilde{V}_6$ as was done for $\tilde{V}_{4Y}$ in Ref. 16. That is, we separate $W'_{\geqslant 6Y}$ into $\tilde{W}'^{1}_{\geqslant 6Y}$ and $\tilde{\tilde{W}}'_{6Y}$,

$$\tilde{\tilde{W}}'_{6Y}(\Psi') \equiv -\frac{\lambda_n^2}{72}\sum_{(\Delta_1,\Delta_2,\Delta_3)}\int_{\Delta_1} dx \int_{\Delta_2} dy\,(L^2\mathscr{T}_{L\Delta_3})_{LxLy}(\psi'_x)^3(\psi'_y)^3$$

and define

$$\tilde{W}'^{2}_{6Y} \equiv -\frac{\lambda_n^2}{72} \sum_{(\Delta_1,\Delta_2)} \int_{\Delta_1} dx \int_{\Delta_2} dy\, (L^2 \mathcal{T}_n)_{LxLy} (\psi'_x)^3 (\psi'_y)^3$$

and

$$\tilde{V}^{n+1}_{\geqslant 6Y}(\Psi')_c \equiv \tilde{W}'^{1}_{\geqslant 6Y}(\xi^{1/2}\Psi') + \tilde{W}'^{2}_{6Y}(\xi^{1/2}\Psi') + \frac{\lambda_{n+1}^2}{72} \sum_{(\Delta_1,\Delta_2)} \int_{\Delta_1} dx \int_{\Delta_2} dy\, (Q_{n+1})_{xy} (\psi'_x)^3 (\psi'_y)^3$$

Then this $\tilde{V}^{n+1}_{\geqslant 6Y}$ satisfies the desired bound B4.

Remark 2. To take the difference of $g_X^{n+1,D'}$, it is more convenient to use the following $\tilde{\rho}$, $\bar{\bar{g}}$, instead of ρ, $\bar{g}$ of Ref. 16:

$$\tilde{\rho}^{D'}_{X'} \equiv \rho^{D'}_{X'} \exp\left\{ \sum_{\Delta \subset X' \cap D'} [W'_\Delta(0) + W'_{2\Delta}(\Psi')] \right\}$$

$$\begin{aligned} \bar{\bar{g}}^{D'}_X \equiv & \sum_{\{X'_\gamma\}} \sum_{\{Y_\alpha\}, Y_\alpha > \Delta} \sum_{\{Y_\beta\}} \prod_\gamma \tilde{\rho}^{D'}_{X'_\gamma} \prod_\alpha \{\exp[W'_{Y_\alpha}(0) + W'_{2Y_\alpha}] - 1\} \\ & \times \prod_\beta [\exp(\tilde{\tilde{W}}'_{4Y_\beta} + \tilde{\tilde{W}}'_{6Y_\beta}) - 1] \\ & \times \exp\left\{ - \sum_{\Delta \subset X \setminus D'} W'_{\geqslant 4\Delta} + \frac{\lambda}{4!} \int_{X \setminus D'} (\psi')^4 \right. \\ & \left. + \sum_i \sum_{\Delta < Y \subset \hat{D}'_i} [W'_Y(0) + W'_{2Y}] \right\} \end{aligned}$$

We thus make full use of expected cancellations between large-field contributions, and take the difference efficiently.

4. TRAJECTORY IN REGION II

Here we investigate the noncritical trajectory not too close to the critical one.

In Section 3 [or Theorem 2.2(i)], we have done the iteration until $\Delta\mu_n$ grows to $(n_0+n)^{-1}$. Because $|(\mu_n)_c| \leqslant (n_0+n)^{-3/2}$, this means (originally $\mu > \mu_c$)

$$(\mu_n)_{nc} \geqslant (n_0+n)^{-1} - (n_0+n)^{-3/2} \geqslant (9/10)(n_0+n)^{-1} \tag{4.1}$$

This is sufficiently massive for us to perform the iteration without referring to the critical theory. In this section, we will consider the noncritical theory only, so we will omit the subscript *nc* in the following.

First, we consider the effective Hamiltonian at $n = n_1 + 1$.

Lemma 4.1. Consider the BST fom n_1 to n_1+1. The $\mathscr{H}^{n_1+1}$ satisfies D_{n_1+1} and E_{n_1+1}, except for E1, with the following recursion relations:

$$\tilde{\mu}' = L^2\mu + \delta\mu^4, \qquad |\delta\mu^4| \leqslant (n_0+n)^{-3/2}$$
$$\text{R1} \quad \text{for } \lambda \to \lambda' \tag{4.2}$$

Proof of Theorem 2.2(ii) for $n = n_1 + 1$, assuming Lemma 4.1. This is almost obvious. We omit the details. ∎

Now for $n \geqslant n_1+2$, we use the folowing inductive proposition.

Proposition 4.2. Suppose $\mathscr{H}^n$ satisfies D_n and E_n. Then $\mathscr{H}^{n+1}$ satisfies D_{n+1} and E_{n+1}, except for E1_{n+1}. Here the recursion $\text{RM}_{n\to n+1}$ also holds.

Proof of Theorem 2.2(ii), Assuming Proposition 4.2. First note that the proposition itself iterates only if we can prove, in addition, the bounds E1_{n+1}.

Case a. The bound on $\tilde{\mu}_{n+1}$.

We use the recursion relation RM for $\tilde{\mu}$ with the bound on $\tilde{\mu}_n$, $|\tilde{\mu}_n| \geqslant (n_0+n)^{-1}$, yielding

$$\tilde{\mu}_{n+1} \geqslant L^2(n_0+n)^{-1} - (n_0+n)^{-13/8} \geqslant (n_0+n+1)^{-1}$$

Case b. The bound on λ_{n+1}.

We write $n = n_1 + m$. First note that as $\tilde{\mu} > 0$, $I^{(\tilde{\mu})} \leqslant I_+$.

The recursion relation RM, $\lambda_l \to \lambda_{l+1}$ for $n_1 \leqslant l \leqslant n_1 + n$, is

$$\lambda' = \lambda[1 - \lambda(3I^{(\tilde{\mu}_n)}/2 - \delta\lambda^3)]$$

Now take the inverse of both sides using $1 + x \leqslant (1-x)^{-1} \leqslant 1 + 2|x|$ (for $|x| \leqslant 1/2$), and take the summation over l:

$$(\lambda_{n_1+m+1})^{-1} - (\lambda_{n_1})^{-1} \leqslant \sum_{l=n_1}^{n_1+m} |3I_l^{(\tilde{\mu}_l)} + 2\delta\lambda^3|$$
$$\geqslant \sum_{l=n_1}^{n_1+m} \delta\lambda^3$$

Estimating the summation by the bounds on I_l and $\delta\lambda^3$, we obtain

$$\lambda_{n+1} = \lambda_{n_1+m+1} \leqslant [(\lambda_{n_1})^{-1} - 2(n_0+n_1)^{8/9}]^{-1}$$
$$\leqslant \frac{C_+}{n_0 + n_1 - 2C_+(n_0+n_1)^{8/9}} \tag{4.3}$$

This yields the desired upper bound as long as, e.g., $m \leqslant (9/10)(n_0+n_1)$. The lower bound can be obtained in a similar (in fact easier) way. ∎

Lemma 4.1 is proven in Sections 4.1–4.4. For this, we assume that the reader is familiar with the method of Gawędzki and Kupiainen. We first assume $0<\tilde{\mu}'<L^5(n_0+n)^{-1}$, and then check that we can in fact choose such $\tilde{\mu}'$.

Proposition 4.2 can be proven in the same way as Lemma 4.1, so we will omit its proof.

4.1. Expression of the Effective Hamiltonian

Let us first recall the formulation of Gawędzki and Kupiainen, which we used in Section 3. We have the relation [(2.30) of Ref. 16]

$$d\mu_{G_n}(\Phi^n) = d\mu_{\zeta^{-1}G_{n+1}}(\Phi^{n+1}) \times d\mu_1(Z^n) \tag{4.4}$$

This, combined with the expression of the nth effective Hamiltonian

$$d\Phi^n \cdot e^{-\mathscr{H}^n(\mathbf{\Phi}^n)} = d\mu_{G_n}(\mathbf{\Phi}^n)\cdot e^{-V^n(\mathbf{\Psi}^n)}\big|_{\mathbf{\Psi}=\mathscr{A}_n\mathbf{\Phi}^n} \tag{4.5}$$

led us to the formula (3.1). In (3.1) the $(n+1)$th effective potential was given by $V^{n+1}(\mathbf{\Psi}^{n+1})$, $\mathbf{\Psi}^{n+1}=\mathscr{A}_{n+1}\mathbf{\Phi}^{n+1}$ (see note added).

On the other hand, here we want to obtain the $(n+1)$th potential as $V^{n+1}(\mathbf{\Psi}^{n+1})$, but $\mathbf{\Psi}^{n+1}=\mathscr{A}^{(\tilde{\mu}')}_{n+1}\mathbf{\Phi}^{n+1}$. Moreover, we want to express the Gaussian part in terms of $G^{(\tilde{\mu}')}_{n+1}$ rather than G_{n+1}, and in addition, $\exp(-\mathscr{H}^{n+1})$ must be analytic on $\mathscr{D}^{(\mu')}(D, L^{-n}\Lambda)$.

For this purpose, we use the relation of Appendix B, Proposition B.2. Here it reads

$$\begin{aligned}&(\mathbf{\Phi}^{n+1}, (G_{n+1})^{-1}\mathbf{\Phi}^{n+1})\\ &\quad = (\mathbf{\Phi}^{n+1}, (G^{(\tilde{\mu}')}_{n+1})^{-1}\mathbf{\Phi}^{n+1})\\ &\qquad - \tilde{\mu}'^2(\Phi^{n+1}, (\delta I), \Phi^{n+1}) - \tilde{\mu}'\int dx\, (\psi'_x)^2\big|_{\mathbf{\Psi}'=\mathscr{A}^{(\tilde{\mu})}\mathbf{\Phi}^{n+1}}\end{aligned} \tag{4.6}$$

and the kernel δI satisfies

$$|(\delta I)_{xy}| \leqslant \mathrm{const}\cdot \exp[-\beta|x-y|] \tag{4.6$'$}$$

Substituting this into G_{n+1} of (4.4), we obtain

$$d\mu_{G_n}(\mathbf{\Phi}^n) = d\mu_{G^{(\tilde{\mu}')}_{n+1}}(\mathbf{\Phi}^{n+1}) \times d\mu_1(Z^n) \times e^{-\delta E} \tag{4.7}$$

$$\begin{aligned}\delta E &= \tfrac{1}{2}(\zeta_n - 1)(\Phi^{n+1}, (G^{(\tilde{\mu}')}_{n+1})^{-1}\mathbf{\Phi}^{n+1})\\ &\quad - \tfrac{1}{2}\zeta_n\, \tilde{\mu}'^2(\Phi^{n+1}, (\delta I), \Phi^{n+1}) - \tfrac{1}{2}\zeta_n\, \tilde{\mu}' \int dx\, (\psi'_x)^2\big|_{\mathbf{\Psi}'=\mathscr{A}^{(\tilde{\mu}')}\mathbf{\Phi}^{n+1}}\end{aligned} \tag{4.7$'$}$$

Equations (4.7) and (4.5) give (see note added)

$$\begin{aligned} d\Phi^{n+1} &\exp[-\mathscr{H}^{n+1}(\mathbf{\Phi}^{n+1})] \\ &= d\mu_{G_{n+1}^{(\bar{\mu}')}}(\mathbf{\Phi}^{n+1}) \exp(-\delta E) \\ &\quad \times \int d\mu_1(Z) \{\exp[-V^n(\mathbf{\Psi}^n = \zeta_n^{1/2} L^{-1} \mathbf{\Psi}^{n+1} + \mathscr{Z})]\} \\ &\quad \times (\mathbf{\Psi}^{n+1} = 0)^{-1}|_{\mathbf{\Psi}^{n+1} = \mathscr{A}_{n+1}\mathbf{\Phi}^{n+1}} \end{aligned} \tag{4.8}$$

We still have to rewrite

$$\mathbf{\Psi}^{n+1} = \mathscr{A}_{n+1} \mathbf{\Phi}^{n+1}$$

in V^n in favor of

$$\mathbf{\Psi}' = \mathscr{A}_{n+1}^{(\bar{\mu}')} \mathbf{\Phi}^{n+1}$$

Recall that $\exp[-\mathscr{H}(\mathbf{\Psi})]$ had a physical maning *only when* $\mathbf{\Psi} = \mathscr{A}\mathbf{\Phi}$. (We extended $\mathbf{\Psi}$ to be complex in order to perform BST.) Taking this into account, we define $\mathbf{\Psi}^{n+1}$ as a function of $\mathbf{\Psi}'$:

$$\begin{aligned} \psi_x^{n+1} &\equiv \psi'_x + \int dy\, \mathscr{D}_{xy} \psi'_y \\ \mathscr{D}_{xy} &\equiv \mathscr{A}_{x[y]} - \mathscr{A}_{x[y]}^{(\bar{\mu}')} \end{aligned} \tag{4.9}$$

where $[y]$ denotes the closest integer to y. Then we can rewrite (4.8) as

$$\begin{aligned} d\mathbf{\Phi}^{n+1} &\exp[-\mathscr{H}^{n+1}(\mathbf{\Phi}^{n+1})] \\ &= d\mu_{G_{n+1}^{(\bar{\mu}')}}(\mathbf{\Phi}^{n+1}) \exp(-\delta E) \\ &\quad \times \int d\mu_1(Z) (\exp[-V^n(\mathbf{\Psi}^n = \zeta_n^{1/2} L^{-1}\{\mathbf{\Psi}' + \mathscr{D}\mathbf{\Psi}'\} + \mathscr{Z})]) \\ &\quad \times (\mathbf{\Psi}' = 0)^{-1}|_{\mathbf{\Psi}' = \mathscr{A}_{n+1}^{(\bar{\mu}')}\mathbf{\Phi}^{n+1}} \end{aligned} \tag{4.10}$$

Now, as in Section 11 of Ref. 16, we extend δE (until now defined only for $\mathbf{\Psi} = \mathscr{A}\mathbf{\Phi}$) to a suitable analytic function of $\mathbf{\Psi}'$, and for

$$\mathbf{\Psi}' \in \mathscr{D}^{(\bar{\mu}')}(D, L^{-(n+1)}\Lambda) \tag{4.11}$$

define

$$\exp[-W^n(\mathbf{\Psi}')] \equiv \int d\mu_1(Z) \exp[-V^n(L^{-1}\{\mathbf{\Psi}' + \mathscr{D}\mathbf{\Psi}'\} + \mathscr{Z})] \tag{4.12}$$

$$\begin{aligned} \exp[-V^{n+1}(\mathbf{\Psi}')] &\equiv \exp[-\delta E(\mathbf{\Psi}')] \\ &\quad \times \exp[-W^n(\zeta^{1/2}\mathbf{\Psi}')]/\exp[-W^n(0)] \end{aligned} \tag{4.13}$$

Then, formally, $\exp[-V^{n+1}(\Psi')]$ is analytic on (4.11) and satisfies all the desired properties.

This is the desired expression of $\exp(-\mathscr{H}^{n+1})$. Now we have to estimate $\exp[-W''(\Psi')]$, rewrite δE, and turn them into the form of assumption D_{n+1}.

4.2. Decoupling Expansion

To estimate (4.12), we perform a decoupling expansion similar to that of Gawędzki and Kupiainen (Ref. 16, Section 5).

1. Localize the regions where $\mathscr{Z}$ is large.
2. Mayer-expand $\tilde{V}_{2Y}$ and $\tilde{V}_{\geqslant 4Y}$.
3. Decouple the nonlocality caused by the kernel $\mathscr{M}$, relate $\mathscr{Z}$ and Z.

3′. But here is one more source of nonlocality: $\mathscr{D} \equiv \mathscr{A} - \mathscr{A}^{(\bar{\mu}')}$. To decouple this, we introduce the parameter t (as well as s) and define [cf. Ref. 16, (5.12)][2]

$$(\mathscr{D}^t)_{xy} \equiv \begin{cases} \mathscr{D}_{xy}, & x, y \in U_k \\ t\mathscr{D}_{xy}, & x \in U_k, \quad y \in U_{k'} \end{cases}$$

and use similar interpolation formulas.

3″. As a result, we have [cf. Ref. 16, (5.23)]

$$\begin{aligned} &\exp[-W''(\Psi')] \\ &\quad = \sum_{\bar{p}} \sum_{\{X_i\}} \sum_{\{Y_\alpha\}} \sum_{\{Y_\beta\}} \sum_{\{\mathscr{U}_\gamma\}} \int \prod_\gamma ST(\mathscr{U}_\gamma) \prod_i g_{X_i}^{nD}(\Psi^{st}) \\ &\quad\quad \times \exp\left[- \int dx \, (\tfrac{1}{2}\mu - \tfrac{1}{4}\lambda \mathscr{G}_{xx})(\psi_x^{st})^2 - \sum_i \sum_{Y \subset X_i} \tilde{V}_{2Y}(\Psi^{st}) \right. \\ &\quad\quad \left. - \frac{\lambda}{4!} \int_{\sim D} dx \, (\psi_x^{st})^4 \right] \equiv \prod\nolimits_\alpha [\exp(-\tilde{V}_{2Y_\alpha}) - 1] \\ &\quad\quad \times \prod_\beta [\exp(-\tilde{V}_{\geqslant 4Y_\beta}) - 1] \, \chi_{\bar{p}}(\Phi) \, d\mu_1(\Phi) \\ &\quad\quad\quad\quad \Psi^{st} \equiv L^{-1}(\Psi' + \mathscr{D}^t\Psi') + \mathscr{Z}^s \end{aligned}$$

Here $\{\mathscr{U}_\gamma\}$ is a partition of $\{U_k\}$, and

$$ST(\mathscr{U}) = \sum_{\Gamma_s, \Gamma_t} \int dt_{\Gamma_t} \, ds_{\Gamma_s} \, \partial_{t_{\Gamma_t}} \partial_{s_{\Gamma_s}}$$

[2] This t should not be confused with the temperature, $t \equiv \mu - \mu_c$.

where the summation over Γ_t and Γ_s runs over all t- and s-graphs such that $\Gamma_t \cup \Gamma_s$ connects all $U_k \in \mathscr{U}$.

4. Separating the contribution from $Y = \Delta$, and turning this into a system of disjoint polymers, we have

$$\exp[-W''(\Psi')] = \sum_{\{X'_\gamma\}} \prod_\gamma \rho^{D'}_{X'_\gamma}(\Psi') \exp\left[-\sum_{\Delta \subset \sim D'} W''_\Delta(\Psi')\right]$$

$$W''_\Delta(\Psi') \equiv [(5.26) \text{ of Ref. 16, with } \Psi^0 \equiv L^{-1}(\Psi' + \mathscr{D}^0\Psi') + \mathscr{X}^0]$$

$$\begin{aligned}\rho^{D'}_{X'}(\Psi') \equiv & \sum_{\bar{p}} \sum_{\{X_i\}} \sum_{\{Y_\alpha\}} \sum_{\{Y_\beta\}} \int ST(\mathscr{U}) \left[\prod_i g^{nD}_{X_i}(\Psi^{st})\right] \\ & \times \exp\left[-\int_{LX'} dx\, (\tfrac{1}{2}\mu - \tfrac{1}{4}\lambda \mathscr{G}_{xx})(\psi^{st}_x)^2 - \sum_i \sum_{Y \subset X_i} \tilde{V}_{2Y}(\Psi^{st}) \right.\\ & \left. - \frac{\lambda}{4!}\int_{LX'} dx\, (\psi^{st}_x)^4\right] \prod_\alpha [\exp(-\tilde{V}_{2Y_\alpha}) - 1] \\ & \times \prod_\beta [\exp(-\tilde{V}_{\geqslant 4Y_\beta}) - 1] \\ & \times \chi_{\bar{p}}(Z_{LX'})\, d\mu_1(Z_{LX'}) \exp\left[\sum_{\Delta \subset X' \setminus D'} W''_\Delta(\Psi')\right]\end{aligned}$$

(The summations are almost the same as (5.25) of Ref. 16, except that LX' is connected by a graph made of s- and t-lines.)

This is the desired decoupling expansion.

4.3. Evaluation of W''_Δ and Polymer Activities

First let us evaluate W''_Δ. This is done in almost the same way as in the case of W'_Δ (Ref. 16, Section 7). The only difference is the appearance of $\mathscr{D}^0$ in the definition of Ψ^0. But since we are expecting $|\tilde{\mu}'| \leqslant L^5(n_0+n)^{-1}$, $|\mathscr{D}^0| \leqslant O(1)(n_0+n)^{-1}$, and the contribution from $\mathscr{D}^0$ is quite small. The result is

$$\begin{aligned} W''_\Delta(\Psi') = & [W'_\Delta(\Psi') \text{ of (7.22) of Ref. 16}] \\ & + \tfrac{1}{6}\int_\Delta dx \int_\Delta dy\, \mathscr{D}^0_{yx} \psi'_x (\psi'_y)^3 \\ & - \tfrac{1}{6}\lambda L^2 \mu \int_\Delta dx \int_\Delta dy\, (L^2 \mathscr{T}_{L\Delta})_{Lx Ly}\, \psi'_x(\psi'_y)^3 \end{aligned}$$

Now let us turn to the evaluation of ρ_X. In this case, the contribution from $\mathscr{D}^t$ is also small, and is treated by the Cauchy estimates. We allow complex values for t,

$$|t_{kk'}| \leqslant (n_0+n)^{9/10} \exp[\beta/2\, d(U_k, U_{k'})]$$

Now, since we are expecting $|\tilde{\mu}'| \leqslant L^5(n_0+n)^{-1}$,

$$\Psi^{st} \in \mathscr{D}(LD' \cup R_k, U_k) \qquad \text{for} \quad \Psi' \in \tfrac{1}{2}L\mathscr{D}^{(\tilde{\mu}')}(D, L^{-(n+1)}\Delta)$$

We can thus use the induction bounds for $\mathscr{H}^n$, and proceed as in Section 8 and 10 of Ref. 16. Because we have allowed such large values for t, the Cauchy estimates show that the difference caused by the appearance of t is quite small. The result is

$$\begin{aligned}\rho_X^\phi = {} & [\rho_X^\phi \text{ of (8.24), Ref. 16}] \\ & -\tfrac{1}{6}\lambda \sum_{(\Delta_1,\Delta_2)} \int_{\Delta_1} dx \int_{\Delta_2} dy\, \mathscr{D}_{yx} \psi'_x(\psi'_y)^3 \\ & +\tfrac{1}{6}\lambda\mu L^2 \sum_{(\Delta_1,\Delta_2,\Delta_3)} \int_{\Delta_1} dx \int_{\Delta_2} dy\, (L^2\mathscr{T}_{L\Delta_3})_{LxLy}\, \psi'_x(\psi'_y)^3\end{aligned}$$

$$\begin{aligned}|\tilde{\rho}_{X'}^{D'}(\Psi')| \leqslant \exp\bigg[& (L^4+1)\, C_2 |D' \cap X'| - \frac{L^2}{2}\left(\frac{\lambda_n}{24}\right)^{1/2} \int_{D' \cap X'} dx\, |\psi'_x|^2 \\ & + \lambda_n \int_{D' \cap X'} dx\, (\operatorname{Im} \psi'_x)^4 - 5\alpha \mathscr{L}(x)\bigg]\end{aligned}$$

for $\Psi' \in (L/8)\, \mathscr{D}^{(\tilde{\mu}')}(D', X')$.

4.4. Determination of λ′, μ̃′, and the Large-Field Bound

First let us determine λ'. We proceed as in Section 9 of Ref. 16. The explicit difference between our case and that of Gawędzki and Kupiainen resides in only the following two terms:

$$\begin{aligned} & -\tfrac{1}{6}\lambda \sum_{(\Delta_1,\Delta_2)} \int_{\Delta_1} dx \int_{\Delta_2} dy\, \mathscr{D}_{yx} \psi'_x(\psi'_y)^3 \\ & \qquad +\tfrac{1}{6}\lambda\mu L^2 \sum_{(\Delta_1,\Delta_2,\Delta_3)} \int_{\Delta_1} dx \int_{\Delta_2} dy\, (L^2\mathscr{T}_{L\Delta_3})_{LxLy}\, \psi'_x(\psi'_y)^3\end{aligned}$$

But, because

$$\int_{\Delta_2} dy\, \mathscr{D}_{yx} = \int_{\Delta_2} dy\, [\mathscr{A}_{y[x]} - \mathscr{A}^{(\mu')}_{y[x]}] = \sum_{y \in \Delta_2} (\delta_{y[x]} - \delta_{y[x]}) = 0$$

the first term can be rewritten as

$$-\tfrac{1}{6}\lambda \sum_{(\Delta_1,\Delta_2)} \int_{\Delta_1} dx \int_{\Delta_2} dy\, \mathscr{D}_{yx} \psi'_x [(\psi'_y)^3 - (\psi'_x)^3]$$

This is exactly of the form of $\tilde{V}_4$ (the fourth variable enters as "difference") and thus has no influence on λ'.

The second term can be treated similarly; because

$$\int_{\Delta_1} dy\, (\mathscr{T}_{L\Delta_2})_{LxLy} = 0$$

we can write it as

$$+\tfrac{1}{6}\lambda\mu L^2 \sum_{(\Delta_1,\Delta_2,\Delta_3)} \int_{\Delta_1} dx \int_{\Delta_2} dy\, (L^2 \mathscr{T}_{L\Delta_3})_{LxLy} \psi'_x [(\psi'_y)^3 - (\psi'_x)^3]$$

This is again of the form of $\tilde{V}_4$. We thus have the same recursion relation for λ' as that of the massless case.

Next, let us turn to terms quadratic in $\mathbf{\Psi}'$. They are of the same form as the massless ones, and can be treated as in Section 11 of Ref. 16:

$$W''_2 = \int dx \left[\frac{L^2\mu}{2} - \frac{\lambda_n}{4}(L^2\mathscr{G} - L^2\mathscr{T})_{LxLx}\right](\psi'_x)^2 + \sum_Y (\tilde{W}'^2_{2Y} + \tilde{W}'^2_{2Y})$$

In addition, here we have the δE term. By the result of Appendix B,

$$\delta E = \left[\tfrac{1}{2}(\zeta - 1)\int dx\, (\partial \psi'_x)^2 - \tfrac{1}{2}\tilde{\mu}' \int dx\, (\psi'_x)^2\right]\Bigg|_{\mathbf{\Psi}' = \mathscr{A}^{(\tilde{\mu}')}_{n+1}\mathbf{\Phi}^{n+1}}$$
$$- \tfrac{1}{2}\zeta\tilde{\mu}'^2(\mathbf{\Phi}^{n+1}, \delta I, \mathbf{\Phi}^{n+1})$$

As in Section 11 of Ref. 16, we can extend

$$-(\mathbf{\Phi}^{n+1}, \delta I, \mathbf{\Phi}^{n+1}) + \sum \tilde{W}'_{2Y}$$

to $\mathbf{\Psi}' \in \mathscr{D}^{(\tilde{\mu}')}(D, L^{-(n+1)}\Lambda)$ as

$$\int dx [\tfrac{1}{4}\lambda'\zeta^{-1}(\mathscr{G}^{(\tilde{\mu}')}_{n+1})_{xx} - \tfrac{1}{4}\lambda_n(\mathscr{G}_{n+1})_{xx}](\psi'_x)^2|_{\mathbf{\Psi}' = \mathscr{A}^{(\tilde{\mu}')}_{n+1}\mathbf{\Phi}^{n+1}}$$
$$- \tfrac{1}{2}(\mathbf{\Phi}^{n+1}, \delta I, \mathbf{\Phi}^{n+1}) + \sum_Y \tilde{W}'_{2Y}$$
$$= \tfrac{1}{2}\delta c \int dx\, (\partial_\mu \psi'_x)^2|_{\mathbf{\Psi}' = \mathscr{A}^{(\tilde{\mu}')}_{n+1}\mathbf{\Phi}^{n+1}} + \tfrac{1}{2}\delta\mu \int dx\, (\psi'_x)^2|_{\mathbf{\Psi}' = \mathscr{A}^{(\tilde{\mu}')}_{n+1}\mathbf{\Phi}^{n+1}}$$
$$+ \text{irrelevant terms}$$

and we obtain

$$(\text{quadratic in } \Psi' \text{ in } V^{n+1})$$

$$= \int dx\, (\psi'_x)^2[\tfrac{1}{2}\zeta L^2\mu - \tfrac{1}{4}\lambda'(\mathscr{G}^{(\tilde{\mu}')}_{n+1})_{xx} + \tfrac{1}{2}\zeta\cdot\delta\mu - \tfrac{1}{2}\tilde{\mu}']$$

$$+ \int dx\, (\partial\psi'_x)^2[\tfrac{1}{2}(\zeta - 1 + \zeta\cdot\delta c)] + \text{irrelevant terms}$$

$$|\delta c| \leqslant O(1)(n_0+n)^{-7/4}, \qquad |\delta\mu| \leqslant O(1)(n_0+n)^{-7/4}$$

We choose

$$1-\zeta = \zeta\cdot\delta c, \qquad \zeta\cdot L^2\mu = \tilde{\mu}' - \zeta\cdot\delta\mu$$

and get

$$(\text{quadratic in } \Psi' \text{ in } V^{n+1})$$

$$= -\tfrac{1}{4}\lambda' \int dx\, (\psi'_x)^2(\mathscr{G}^{(\tilde{\mu}')}_{n+1})_{xx} + \text{irrelevant terms}$$

with

$$|\zeta - 1| \leqslant (n_0+n)^{-3/2}$$

and

$$\tilde{\mu}' = \zeta(L^2\mu + \delta\mu) = L^2\mu + \delta\mu^4, \qquad |\delta\mu^4| \leqslant (n_0+n)^{-3/2}$$

Now that we have fixed λ' and $\tilde{\mu}'$, we can proceed to the large-field bound. Since this can be treated in exactly the same way as in Section 10 of Ref. 16, we omit the details.

5. TRAJECTORY IN REGION III

Here we investigate a noncritical trajectory far from the critical one. We will be rather sketchy, because this is an easy exercise of the methods already presented.

We use the following inductive proposition.

Proposition 5.1. Suppose $\mathscr{H}^n$ satisfies F_n and G_n. Then $\mathscr{H}^{n+1}$ satisfies F_{n+1} and G_{n+1}, except for $\mathrm{G1}_{n+1}$. Here the recursion $\mathrm{RN}_{n\to n+1}$ also holds.

Proof of Theorem 2.2(iii), Assuming Proposition 5.1. As in Section 4, the proposition itself iterates only if we can prove, in addition, the bounds $\mathrm{G1}_{n+1}$.

By RN and G1 for $\tilde{\mu}$ ($n_1' \equiv n_1 + \frac{9}{10}(n_0 + n_1)$),

$$\tilde{\mu}_{n+1} \geqslant L^2 \tilde{\mu}_n + \delta\mu^5 \geqslant \tfrac{1}{2}L^2(\tfrac{1}{2}L^2)^{n-n_1'}\tilde{\mu}_{n_1'}$$

Moreover,

$$|\lambda_{n+1} - \lambda_{n_1'}| = \left| \sum_{k=n_1'}^{n} \delta\lambda_k \right| \leqslant 2(n_0 + n_1')^{-3/2}$$

Combining this with the bound E1 on $\lambda_{n_1'}$, we can get the desired bound. ▮

Proof of Proposition 5.1. First note that Assumptions F and G are almost the same as D and E. Thus, along the same line of argument as that of Section 4, we can easily prove Proposition 5.1, but with the recursion RN replaced by RM. Our remaining task is therefore to improve the recursion RM into RN. This is easily done by the following arguments.

1. Estimation of W_Δ''. Contributions to $\mathbf{\Psi}'^2$ and $\mathbf{\Psi}'^4$ from W_Δ'' except for

$$-\int dx \left[\frac{\lambda}{4!} \mathbf{\Psi}'^4 - \frac{\lambda}{4} \mathscr{G}'_{xx} \mathbf{\Psi}'^2 \right]$$

contain at least one $\mathscr{X}$–$\mathscr{X}$ contraction, and are of order (see Proposition A.3)

$$|\mathscr{T}^{(\mu_n)}| \sim \mu_n^{-1} \ll (2/L)^{2(n-n_1')}$$

2. Estimation of $\rho_{X'}$. First change the definition of $\chi_{\bar{p}}$ as

$$\chi_{\bar{p}} \equiv \prod_u \chi[p_u(n_0 + n_1')^{1/4}(L/2)^{(n-n_1')/2} \leqslant Z_u$$
$$\leqslant (p_u + 1)(n_0 + n_1')^{1/4}(L/2)^{(n-n_1')/2}]$$

Then terms with $\bar{p} \neq 0$ become of the order

$$\exp\left[-(n_0 + n_1')^{1/2}(L/2)^{(n-n_1')} \sum_u p_u^2 \right]$$

On the other hand, because $\mathscr{M}_{xu}^{(\mu)} \leqslant O(\mu^{-1/2}) \exp(-\beta|x-u|)$ and we expect

$$|\delta\mu^5| \leqslant (n_0 + n_1')^{-3/2}(2/L)^{(n-n_1')/2}$$

we can still allow s, t to take such large values as

$$|s_{kk'}| \leqslant (L/2)^{(n-n_1')/2} 2r \exp[\tfrac{1}{2}\beta d(U_k, U_{k'})]$$
$$|t_{kk'}| \leqslant (L/2)^{(n-n_1')/2} (n_0+n_1')^{3/2} \exp[\tfrac{1}{2}\beta d(U_k, U_{k'})]$$

Then each s- and t-derivative provides us extra $(2/L)^{(n-n_1')/2}$ factors compared with the result of Section 4.

Terms (contributing to ψ'^2 or ψ'^4) without s- or t-derivatives are evaluated explicitly, and are also shown to have extra factors $(2/L)^{(n-n_1')/2}$.

3. Summary. As a result, for terms in W_Y'' contributing to ψ'^2 or ψ'^4, we have at least one extra $(2/L)^{(n-n_1')/2}$ factor compared with those of Section 4. We thus have

$$|\delta\mu^5| \leqslant (|\delta\mu| \text{ of Section 4}) \cdot (2/L)^{(n-n_1')/2}$$

and

$$|\delta\lambda^5| \leqslant (|\delta\lambda| \text{ of Section 4}) \cdot (2/L)^{(n-n_1')/2}$$

and the proposition is proved. ∎

6. REFINED BOUNDS ON THE MOTION OF λ_n AND μ_n

We have proven a rough picture on the trajectory in Sections 3–5. Now we can refine the result and derive more detailed information on the motion of $\mathscr{H}_n$.

6.1. Bounds on n_1

Proof of Theorem 2.3(i). This follows easily from the rought bounds (2.3) on the motion of $\Delta\mu_n$. Combined with the definition of n_1, they yield

$$t(2L^2)^{n_1} \geqslant |\Delta\mu_{n_1}| \geqslant (n_0+n_1)^{-1} \tag{6.1}$$

$$t(L^2/2)^{n_1-1} \leqslant |\Delta\mu_{n_1-1}| \leqslant (n_0+n_1-1)^{-1} \leqslant (n_0)^{-1} \tag{6.2}$$

To get the lower bound, we subtract $\ln(n_0)$ from both sides of (6.1):

$$n_1 \ln(2L^2) + \ln[(n_0+n_1)/n_0] \geqslant |\ln t| - \ln n_0$$

Since $\ln(1+x) \leqslant x$ (for $x \geqslant 0$), we have

$$n_1 \geqslant \frac{|\ln t| - \ln n_0}{\ln(2L^2) + 1/n_0} \geqslant \frac{|\ln t| - \ln n_0}{2\ln L + 1}$$

The upper bound can be obtained in a similar way. ∎

6.2. Refined Bounds on $\Delta\mu_n$

Theorem 2.3(ii) is proved by the following two lemmas.

Lemma 6.1. Consider the case when $|t| \leqslant (10n_0)^{-1}$. Define

$$S_n \equiv \sum_{k=0}^{n-1} I_k, \qquad T_n \equiv \sum_{k=1}^{n-1} I_k/S_k \tag{6.3}$$

Then (i) $(\lambda_n)_{\text{critical}}$ satisfies

$$\lambda_n^{-1} \begin{cases} \leqslant \lambda_0^{-1} + \frac{3}{2}S_n + 4[(n_0+n)^{8/9} - (n_0)^{8/9}] \\ \geqslant \lambda_0^{-1} + \frac{3}{2}S_n - 3[(n_0+n)^{8/9} - (n_0)^{8/9}] \end{cases} \tag{6.4}$$

(ii) $\Delta\mu_n$ satisfies (for $n \leqslant n_1$)

$$\Delta\mu_n = tL^{2n} \exp\{-F_n\} \tag{6.5}$$

with

$$\left| F_n - \frac{1}{2} \sum_{k=0}^{n-1} \lambda_k I_k \right| \leqslant 12(n_0)^{-1} \tag{6.6a}$$

and

$$|F_n - \tfrac{1}{3}T_n| \leqslant C + C' \ln n_0 \tag{6.6b}$$

where C and C' are positive constants depending on L.

Lemma 6.2. For $n \leqslant n_1$,

$$nI_- \leqslant S_n \leqslant nI_+ \tag{6.7}$$

For $17 \leqslant n \leqslant n_1$, we have (i)

$$nI_{n_1} + 15I_- - 16I_{n_1} \leqslant S_n \leqslant nI_{n_1} + 15I_+ - 14I_{n_1} \tag{6.8}$$

and (ii)

$$n - 16 + 15I_-/I_+ \leqslant S_n/I_n \leqslant n - 14 + 15I_+/I_- \tag{6.9}$$

and (iii)

$$T_n \begin{cases} \leqslant \ln(n-16) + 109R + \ln(R/15) \\ \geqslant \ln(n - 14 + 15R) + 109/R + \ln(3 + 15R) \end{cases} \tag{6.10}$$

where $1 \leqslant R \equiv I_+/I_- < \infty$.

Theorem 2.3(ii) is an immediate consequence of Lemmas 6.1 and 6.2.

Proof of Lemma 6.1. (i) Taking the inverse of both sides of R1, we have

$$\lambda'^{-1} = \lambda^{-1}[1 + \lambda(-3I/2 + \delta\lambda^1)]^{-1}$$

Since I is of order 1, and $\delta\lambda^1$ of order $(n_0+n)^{-1/9}$, we can use

$$1 + x \leqslant \frac{1}{1-x} \leqslant 1 + x + 2x^2 \qquad \text{for} \quad 0 \leqslant x \leqslant 1/2$$

to obtain

$$\lambda^{-1} + 3I/2 - \delta\lambda^1 \leqslant \lambda' \leqslant \lambda^{-1} + 3I/2 - \delta\lambda^1 + \lambda^2(2I_+)^2$$

Summing these inequalities from $n = 0$ to $n - 1$, we obtain

$$\lambda_n^{-1} - \lambda_0^{-1} \geqslant \sum_{k=0}^{n-1} (3I_k/2 - \delta\lambda^1)$$

$$\geqslant 3S_n/2 - 3[(n_0+n)^{8/9} - (n_0)^{8/9}]$$

where we used the bound $|\delta\lambda_k^1| \leqslant (n_0+k)^{-1/9}$.

The upper bound is obtained similarly. ∎

(ii) Multiplying the recursion for $\Delta\mu_k$, $0 \leqslant k \leqslant n-1$, we obtain

$$\frac{\Delta\mu_n}{t} = \prod_{k=0}^{n-1} \frac{\Delta\mu_{k+1}}{\Delta\mu_k} = L^{2n} \prod_{k=0}^{n-1} \left(1 - \frac{\lambda_k I_k}{2} + \delta\mu^2\right)$$
$$\equiv L^{2n} \exp(-F_n)$$

Using $-x - x^2 \leqslant \ln(1-x) \leqslant -x$ for $x \leqslant 1/2$, we obtain

$$-F_n = \sum_{k=0}^{n-1} \ln(1 - \lambda_k I_k/2 + \delta\mu^2/2)$$

$$\begin{cases} \leqslant 1/2 \sum_{k=0}^{n-1} [-\lambda_k I_k + (n_0+k)^{-10/9} + (n_0+k)^{-1/6}|\Delta\mu|^{2/3}] \\ \geqslant 1/2 \sum_{k=0}^{n-1} [-\lambda_k I_k - 2(n_0+k)^{-10/9} - (n_0+k)^{-1/6}|\Delta\mu|^{2/3}] \end{cases}$$

By the rough bound on $\Delta\mu$,

$$|\Delta\mu_k| \leqslant |\Delta\mu_{n_1-1}|(L^2/2)^{k-n_1+1}$$
$$\leqslant (n_0+n_1-1)^{-1}(L^2/2)^{k-n_1+1}$$

and

$$\sum_{k=0}^{n-1} (n_0+k)^{-1/6} |\Delta\mu_k|^{2/3}$$

$$\leqslant (n^0)^{-1/6}(n_0+n_1-1)^{-2/3} \sum_{k=0}^{n-1} (L^2/2)^{2(k-n_1+1)/3}$$

$$\leqslant 2(n_0)^{-5/6}$$

It we estimate other summations over k similarly, we obtain (6.6a).

To prove (6.6b), we have to estimate the difference between $\sum \lambda_k I_k/2$ and $T_n/3$. First,

$$\left|\sum_{k=0}^{n-1} \lambda_k I_k/2 - T_n/3\right|$$

$$= \left|\sum_{k=1}^{n-1} [\lambda_k/2 - (3S_k)^{-1}] I_k + \lambda_k I_k/2\right|$$

$$\leqslant I_+ \left|\sum_{k=1}^{n-1} [\lambda_k/2 - (3S_k)^{-1}]\right| + \tfrac{1}{2}(n_0)^{-1/2}$$

Then use the bound (6.4) on λ_k, and estimate the summation. ∎

Proof of Lemma 6.2. We use the uniform bound $I_- \leqslant I_n \leqslant I_+$ together with the limiting property of I_n,

$$\exp(-2L^{7-n/2}) \leqslant I_n/I_{n_1} \leqslant \exp(2L^{7-n/2})$$

which is proven in Appendix C.

Condition (6.7) is immediate from the definition of S_n.

(i) $$S_n \leqslant \sum_{k=0}^{14} I_k + \sum_{k=15}^{n-1} I_k \leqslant 15 I_+ + \sum_{k=15}^{n-1} I_{n_1} \exp(2L^{7-k/2})$$

$$\leqslant 15 I_+ + (n-14) I_{n_1}$$

Similarly for the lower bound.

(ii) We have

$$S_n/I_n = \sum_{k=0}^{n-1} I_k/I_n \leqslant \sum_{k=0}^{14} I_+/I_- + \sum_{k=15}^{n-1} \exp(2L^{7-k/2})$$

and (iii)

$$T_n = \sum_{k=1}^{n-1} (S_k/I_k)^{-1} = \sum_1^{15}(\cdots) + \sum_{16}^{n-1}(\cdots)$$

Use the bound (ii) and estimate the summation. ∎

6.3. Bounds in Region II

To trace the trajectory in region II, we use the following lemma.

Lemma 6.3. Let $0 < t < (n_0)^{-1}$. Then for $0 \leqslant m \leqslant \frac{9}{10}(n_0 + n_1)$,

$$\frac{99}{100} L^{2m} \leqslant \frac{\tilde{\mu}_{n_1+m}}{\tilde{\mu}_{n_1}} \leqslant \frac{101}{100} L^{2m}$$

Theorem 2.3(iii) is a direct consequence of this lemma and the rough bound on $\Delta\mu_{n_1}$.

Proof of Lemma 6.3. By the recursion for $\tilde{\mu}$, we can easily derive the rough bound:

$$(L^2/2)^m \leqslant \tilde{\mu}_{n_1+m}/\tilde{\mu}_{n_1} \leqslant (2L^2)^m$$

and the expression $\displaystyle \frac{\tilde{\mu}_{n_1+m}}{\tilde{\mu}_{n_1}} = L^{2m} \prod_{k=0}^{m-1} \left(1 - \frac{\delta\mu^3}{L^2\tilde{\mu}_{n_1+k}}\right)$

Estimating the product by taking the logarithm, with the bound

$$|\delta\mu^3/L^2\tilde{\mu}| \leqslant (n_0 + n_1)(n_0 + n)^{-3/2}(L^2/2)^{n_1 - n}$$

we can derive the lemma. ∎

7. THE INFINITE-VOLUME LIMIT OF $\mu_c(\lambda; \Lambda_0)$

Proof of Theorem 2.5. Take two tori Λ^1 and Λ^2 of side L^{N_1} and L^{N_2}, respectively (for definiteness we take $N_1 < N_2$), and apply n BSTs $(n < N_1 - N_0)$ to φ^4 systems defined on Λ^1 and Λ^2, respectively. We abbreviate

$$\mu^1 \equiv \mu_c(\lambda; \Lambda^1), \qquad \mu^2 \equiv \mu_c(\lambda; \Lambda^2)$$

We also denote by μ^i_{n,Λ^j} the coefficient μ_n obtained by applying n BSTs to φ^4 system on Λ^j with $\mu_0 = \mu^i$. By definition of $\mu_c(\lambda; \Lambda)$,

$$(\mu^i)_{n,\Lambda^i} \in [-(n_0 + n)^{-3/2}, (n_0 + n)^{-3/2}] \equiv I_n \tag{7.1}$$

for $n \leqslant N_1 - N_0$ and $n \leqslant N_2 - N_0$, respectively. Now consider the identity

$$|\mu^1_{n,\Lambda^1} - \mu^2_{n,\Lambda^2}| = |\mu^1_{n,\Lambda^1} - \mu^1_{n,\Lambda^2} + \mu^1_{n,\Lambda^2} - \mu^2_{n,\Lambda^2}|$$

Fix n sufficiently large, and choose N_1 sufficiently large (compared with n) so as fulfill

$$|\mu^1_{n,\Lambda^1} - \mu^1_{n,\Lambda^2}| \leqslant (n_0 + n)^{-3/2}$$

(That we can choose such N_1 is a consequence of the analysis of Gawędzki and Kupiainen. See Ref. 16, Section 12.) Then for (7.1) to hold, we must have

$$\begin{aligned} |\mu^1_{n,\Lambda^2} - \mu^2_{n,\Lambda^2}| &\leqslant |\mu^1_{n,\Lambda^1} - \mu^2_{n,\Lambda^2}| + |\mu^1_{n,\Lambda^1} - \mu^1_{n,\Lambda^2}| \\ &\leqslant 3(n_0 + n)^{-3/2} \end{aligned} \tag{7.2}$$

On the other hand, for the difference of the LHS, we have [see (2.3)],

$$|\mu^1_{n,\Lambda^2} - \mu^2_{n,\Lambda^2}| \geqslant (L^2/2)^n\, |\mu^1 - \mu^2|$$

Thus, for any n,

$$|\mu_c(\Lambda^1) - \mu_c(\Lambda^2)| \leqslant 3(2L^{-2})^n (n_0 + n)^{-3/2} \tag{7.3}$$

holds for N_1, N_2 sufficiently large (depending on n). This immediately means that the existence domain of $\mu_c(\Lambda)$ shrinks to some point in $\mathbf{R}$ when $\Lambda \to \mathbf{Z}^4$. ∎

APPENDIX

In this appendix, we list and prove some of the necessary properties of Gaussian propagators, kernels, and integrals. We first list some formulas and abbreviations concerning the Fourier transformation, on which the analysis heavily depends.

We use the following abbreviations:

$$f_n(p) \equiv \prod_{\nu=1}^{d} \frac{\sin(p_\nu/2)}{L^n \sin(p_\nu/2L^n)}$$

$$g_n(p) \equiv 2L^{2n} \sum_{\nu=1}^{d} [1 - \cos(p_\nu/L^n)]$$

$$\mu^{(\mu_0)}(p) \equiv \mu_0 + 2 \sum_{\nu=1}^{d} (1 - \cos p_\nu)$$

Fourier representations of various kernels are found in Refs. 11 and 13. We here present only that of G_n, on which we will prove somewhat detailed bounds. $G_n^{(\mu;\xi)}$ of infinite volume with infrared regulator ξ is given by

$$(G_n^{(\mu;\xi)})_{0x}^{(\mathrm{iv})} = \int_{[-\pi,\pi)^d} \frac{d^d p}{(2\pi)^d} e^{ipx} \tilde{G}_n^{(\mu;\xi)}(p)$$

$$\tilde{G}_n^{(\mu;\xi)}(p) = \sum_{|t_\nu| < L^n/2} \frac{[f_n(p+2\pi t)]^2}{\mu + \xi \cdot \delta_{p,0} \sum_1^d \{2L^n \sin[(p_\nu + 2\pi t_\nu)/2L^n]\}^2}$$

Bounds on finite-volume G_n are obtained by periodizing $G_n^{(\mathrm{iv})}$. That is,

$$(G_n)_{xy} = \sum_{R \in L^{N-n}\mathbf{Z}^d} (G_n^{(\mathrm{iv})})_{x,y+R}$$

A1. Bounds on Gaussian Kernels

Proposition A.1. For $\mu \geqslant 0$, the massive Gaussian kernels satisfy the same bounds as the massless ones, i.e., (2.42)–(2.46) of Ref. 16.

Proof. These are proven by the direct analyis in momentum space, in eaxtly the same way as in Refs. 11 and 13. ∎

For the difference between two massive Gaussians, we have:

Proposition A.2. Let μ^1, $\mu^2 \geqslant 0$. Then the Fourier transform of $G_n^{(\mu^i)}$ satisfies: for $|\mathrm{Im}\, p_1| \leqslant 4/5$, $(\mathrm{Re}\, p_1, p_2,..., p_d) \in [-\pi, \pi]^{d-1}$,

$$|[\tilde{G}_n^{(\mu^1)}(p)]^{-1} - [\tilde{G}_n^{(\mu^2)}(p)]^{-1}| \leqslant |\mu^1 - \mu^2| \pi^{7d+4} d^2 \tag{A.1}$$

Also, in configuration space,

$$|(\mathscr{A}_n^{(\mu^1)})_{xy} - (\mathscr{A}_n^{(\mu^2)})_{xy}| \leqslant |\mu^1 - \mu^2| \pi^{8d+5} d^{d+3} e^{-\beta|x-y|} \tag{A.2}$$

and similar bounds for the differences of

$$\frac{\mathscr{A}_{xy} - \mathscr{A}_{xy}}{|x-y|} \quad \text{and} \quad \frac{\partial_\mu \mathscr{A}_{xy} - \partial_\mu \mathscr{A}_{xy}}{|x-y|}$$

hold.

Moreover,

$$|(\mathscr{T}_n^{(\mu^1)})_{xy} - (\mathscr{T}_n^{(\mu^2)})_{xy}| \leqslant \mathrm{const} \cdot |\mu^1 - \mu^2| e^{-\beta|x-y|} \tag{A.3}$$

holds.

Proof. This is also straightforward. Use the Fourier representation and, taking all the necessary differences, follow the line of argument of Refs. 11 and 13. ∎

When the "renormalized mass" μ_n is quite large, the following is more useful.

Proposition A.3. Let $\mu > 0$. Then

$$(G_n^{(\mu)})_{0x} \leqslant \min\{\mu^{-1}, \text{const}(d) \exp[-(\tfrac{1}{2}\mu)^{1/2}(|x|_\infty - 1)]\} \tag{A.4}$$

where const(d) depends only on the lattice dimensionality d, and

$$|(\mathscr{M}^{(\mu)})_{xy}| \leqslant \text{const}(d, L) \cdot \mu^{-1/2} \exp(-\beta |x - y|) \tag{A.5}$$

where const(d, L) depends only on d and L.

Proof. $(G_n^{(\mu)})_{0x} \leqslant \mu^{-1}$ is a direct consequence of the Fourier representation of G_n. The exponentially decaying property is proven by a standard technique of complex translation (see, e.g., Ref. 11). Bounds on $\mathscr{M}$ can be proven similarly. ∎

We will use the following proposition in Part II.

Proposition A.4. Consider $G_n^{(\mu)}$ with

$$\mu \geqslant 100 \tag{A.6}$$

Then (i)

$$\mu_n^{-1}(1 - 2\pi/\sqrt{\mu_n}) \leqslant (G_n^{(\mu)})_{00} \leqslant \mu_n^{-1} \tag{A.7}$$

(ii) For $1 \leqslant |x|_\infty \equiv \max_\mu |x_\mu|$,

$$(G_n^{(\mu)})_{0x} \leqslant 128\mu_n^{-3/2} \exp[-(\mu_n/2)^{1/2}(x_1 - 1 + L^{-n})]$$

and for $x = (x_1, 0, 0, 0)$, $x_1 \geqslant 1$,

$$(G_n^{(\mu)})_{0x} \geqslant 10^{-2}(\mu_n + 3\pi^2)^{-3/2} \times \exp[-(\mu_n + 3\pi^2)^{1/2}(x_1 - 1 + L^{-n})] \tag{A.8}$$

Further (iii)

$$\frac{(G_n^{(\mu)})_{0x}}{(G_n^{(\mu)})_{00}} \leqslant \begin{cases} 1, & |x| = 0 \\ 200\mu_n^{-1/2}, & |x|_\infty = 1 \\ 200\mu_n^{-1/2} \exp[-(\mu_n/2)^{1/2}(|x|_\infty - 1)], & |x|_\infty \geqslant 1 \end{cases} \tag{A.9}$$

Proof. We abbreviate $G_n^{(\mu)}$ as G_n.

(i) $(G_n)_{00}$ is bounded by the direct analysis in the momentum space.

(ii) We first rewrite $G_n^{(\text{iv})}$ by introducing $p'_\nu \equiv p_\nu + 2\pi t_\nu$:

$$(G_n^{(\mu)})_{0x}^{\text{iv}} = \int_{[-\pi L^n, \pi L^n)^{d-1}} \frac{d^{d-1}p}{(2\pi)^{d-1}} \exp(i\mathbf{px}) \times \prod_2^d \left(\frac{\sin(p_\nu/2)}{L^n \sin(p_\nu/2L^n)}\right)^2 F(\mathbf{p}, x_1)$$

$$F(\mathbf{p}, x_1) \equiv \int_{[-\pi L^n, \pi L^n)} \frac{d\theta}{2\pi} \exp(i\theta x_1) \left(\frac{\sin(\theta/2)}{L^n \sin(\theta/2L^n)}\right)^2 \times \frac{1}{B(\mathbf{p}) + [2L^n \sin(\theta/2L^n)]^2}$$

$$B(\mathbf{p}) \equiv \mu + \sum_2^d \left(2L^n \sin \frac{p_\nu}{2L^n}\right)^2$$

Now writing

$$\left(\frac{\sin(\theta/2)}{L^n \sin(\theta/2L^n)}\right)^2 = \exp[i\theta(1 - L^{-n})]\, L^{-2n} \sum_{m,l=0}^{L^n - 1} \exp[-i\theta L^{-n}(m + l)]$$

and using residue calculus, we have (in the complex p_1 plane, contributions from the path $\operatorname{Re} z = \pm\pi L^n$ cancel!)

$$F(\mathbf{p}, x_1) = \frac{1}{2}\left[B\left(1 + \frac{B}{4L^{2n}}\right)\right]^{-1/2} \exp[-\theta_0(x_1 - 1 + L^{-n})] \times \left\{\frac{1 - \exp(-\theta_0)}{L^n[1 - \exp(-L^{-n}\theta_0)]}\right\}^2$$

with

$$\theta_0 \equiv L^n \cosh^{-1}(1 + B/2L^{2n})$$

Now estimating θ_0 and performing the integral over $\mathbf{p}$, we can obtain the upper bounds

$$(G_n^{(\text{iv})})_{0x} \leqslant 4\mu^{-3/2} \exp[-(\mu/2)^{1/2}(x_1 - 1 + L^{-n})]$$

Similarly, for $x=(x_1, 0, 0, 0,...)$,

$$\geqslant \frac{1}{6}\left(\frac{2}{\pi}\right)^{2d-2} \frac{\exp\{-[\mu+(d-1)\pi^2]^{1/2}(x_1-1+L^{-n})\}}{[\mu+(d-1)\pi^2]^{3/2}}$$

[Note that because $F(\mathbf{p}, x_1)$ is nonnegative, so is the integrand. So we can get a lower bound by estimating some suitable part of the integration.]

(iii) This follows immediately from (i) and (ii). ▮

APPENDIX B. RELATIONS BETWEEN Ψ AND Φ

We first recall the result of Gawędzki and Kupiainen.

Proposition B.1. Let $\mu \geqslant 0$. Then

$$(\Phi^n, (G_n^{(\mu)})^{-1}, \Phi^n) \equiv \sum_{x,y\in\Lambda_n} \varphi_x^n (G_n^{(\mu)})_{xy}^{-1} \varphi_y^n$$

$$= \int dx \left[\sum_v (\partial_v \psi_x^n)^2 + \mu(\psi_x^n)^2\right]\Bigg|_{\Psi^n = \mathscr{A}_n^{(\mu)}\Phi^n}$$

Proof. Follows directly from the Fourier representations of G_n and $\mathscr{A}_n$. ▮

We can thus express $(\Phi^n, (G_n^{(\mu)})^{-1}, \Phi^n)$ as a function of Ψ^n restricted to $\mathscr{A}_n \Phi^n$.

Now for the mutual difference between above quantities corresponding to μ^1 and μ^2, we have:

Proposition B.2. Let $\mu^1, \mu^2 \geqslant 0$. Then

$$(\Phi^n, (G_n^{(\mu^1)})^{-1}, \Phi^n)$$

$$= (\Phi^n, (G_n^{(\mu^2)})^{-1}, \Phi^n) + (\mu^1 - \mu^2) \int dx\, (\psi_x^n)^2 \Bigg|_{\Psi^n = \mathscr{A}_n^{(\mu^2)}\Phi^n}$$

$$- (\mu^1 - \mu^2)^2 (\Phi^n, \delta I_n, \Phi^n)$$

with

$$|(\delta I_n)_{xy}| \leqslant \text{const} \cdot e^{-\beta|x-y|}$$

This expresses

$$d\mu_{G_n^{(\mu^1)}}(\Phi^n) \qquad \text{as} \qquad d\mu_{G_n^{(\mu^2)}}(\Phi^n) \times \exp(\text{correction})$$

Proof. By the definition, we have

$$(\Phi^n, (G_n^{(\mu^1)})^{-1}, \Phi^n) - (\Phi^n, (G_n^{(\mu^2)})^{-1}, \Phi^n) - (\mu^1 - \mu^2) \int dx\, (\psi_x^n)^2 \Big|_{\psi^n = \mathscr{A}_n^{(\mu^2)}\Phi^n} = \sum_{y,z \in \Lambda_n} \varphi_y^n I_{yz} \varphi_z^n$$

with

$$I_{yz} \equiv (G_n^{(\mu^1)})_{yz}^{-1} - (G_n^{(\mu^2)})_{yz}^{-1} - (\mu^1 - \mu^2) \int dx\, \mathscr{A}_{n,xy}^{(\mu^2)} \mathscr{A}_{n,xz}^{(\mu^2)}$$

By the Fourier transform,

$$I_{yz} = L^{d(n-N)} \sum_{\substack{p_v \in 2\pi L^{n-N} \mathbf{Z}^d \\ |p_v| < \pi}} e^{ip(y-z)} \tilde{I}(p)$$

with

$$\tilde{I}(p) = [\tilde{G}_n^{(\mu^1)}(p)]^{-1} - [\tilde{G}_n^{(\mu^2)}(p)]^{-1} - (\mu^1 - \mu^2) \sum_{|t_v| < L^n/2} \left[L^{2n} \mu^{(L^{-2n}\mu^2)} \left(\frac{p + 2\pi t}{L^n}\right) \tilde{G}_n^{(\mu^2)}(p) \right]^{-2} \times [f_n(p + 2\pi t)]^2$$

Now by tedious but direct computation, we can rewrite

$$\tilde{I}(p) = (\mu^1 - \mu^2)^2\, \widetilde{\delta I}(p)$$

with

$$\widetilde{\delta I}(p) = [\tilde{G}_n^{(\mu^1)}(p)]^{-1} [\tilde{G}_n^{(\mu^2)}(p)]^{-2} \times \sum_{t,s} \frac{[f_n(p + 2\pi t)\, f_n(p + 2\pi s)]^2}{[\mu^2 + g_n(p + 2\pi t)]^2 [\mu^2 + g_n(p + 2\pi s)]^2} \times \frac{g_n(p + 2\pi t)[g_n(p + 2\pi t) - g_n(p + 2\pi s)]}{[\mu^1 + g_n(p + 2\pi t)][\mu^1 + g_n(p + 2\pi s)]}$$

Now for this $\widetilde{\delta I}$, we can prove:

Lemma B.3. $\widetilde{\delta I}(p)$ is analytic for $|\mathrm{Im}\, p_1| \leqslant \varepsilon_1(d)$, $(\mathrm{Re}\, p_1, p_2, ..., p_d) \in [-\pi, \pi]^d$ ($\varepsilon_1(d)$ is same as ε of Lemma A.5, Ref. 11), and satisfies there:

$$|\widetilde{\delta I}(p)| \leqslant 2^{5+d} \pi^{7d+2} (d+2)^3$$

Proposition B.2 immediately folows from this lemma, by a standard technique of complex translation. ∎

Proof of Lemma B.3. The analyticity of $\widetilde{\delta I}$ is proven in almost the same way as Lemma A.1 or A.5 of Ref. 11. [Because the $s=t=0$ term is absent in the summation, the zeros of $\mu+g_n(p+2\pi t)$ and $\mu+g_n(p+2\pi s)$ never coincide in the denominator. Thus, they are canceled by zeros of $g_n(p+2\pi t)$ or $g_n(p+2\pi s)$ in the numerator.] The bound on $\widetilde{\delta I}$ is proven by establishing various bounds as was done in the proof of Lemma A.1, Ref. 11. ∎

APPENDIX C. PROPERTIES OF I_n

We prove the following proposition.

Proposition C.1. Let $\mu \geqslant 0$. Then for $L \geqslant 100$,

$$0 \leqslant I_n^{(\mu)} \leqslant I_+ \equiv 37/2 + \tfrac{1}{3}\ln L \tag{C.1}$$

For massless I_n, we have more refined bounds

$$10^{-2}(\ln L - 3) \leqslant I_n \leqslant I_+ \tag{C.2}$$

(We can thus choose $I_- = 10^{-2}$ for $L \geqslant 100$.) As for the limit property of massless I_n, we have, for $n \geqslant 15$, $N \geqslant 30 + 2\underline{n}$ and $15 \leqslant n \leqslant \underline{n}$,

$$|\ln(I_n/I_{\underline{n}})| \leqslant 2L^{7-n/2} \tag{C.3}$$

Proof. The proof is carried out by direct but tedious calculations. We use the following abbreviations:

$$\sum_t \equiv \sum_{t \in \mathbf{Z}^4, |t_\nu| < L^n/2}$$

$$\sum_s \equiv \sum_{s \in \mathbf{Z}^4, |s_\nu| < L^{n+1}/2}$$

$$F_n(p;t) \equiv [f_n(p+2\pi t)]^2 [g_n(p+2\pi t)]^{-1}$$

$$G_n(p;t) \equiv [g_n(p+2\pi t)]^{-1}$$

We further abbreviate $F_n(p;t)$ and $F_{n+1}(p;t)$ as $F(t)$ and $F'(t)$, respectively, and similarly for G's.

We first express I_n as a sum of nine terms:

$$\begin{aligned} I_n &= \int_{\square_0} dx \int dy\, \{2[(L^2\mathscr{G}_n)_{Lx\,Ly} - (\mathscr{G}_{n+1})_{xy}](L^{2n+2}G_0)_{L^{n+1}x\,L^{n+1}y} \\ &\quad - [(L^2\mathscr{G}_n)_{Lx\,Ly}]^2 + [(\mathscr{G}_{n+1})_{xy}]^2\} \\ &= 2(I_{1n} + I_{3n} - I_{4n} + I_{5n}) - I_{2n} - I_{6n} + I_{7n} - I_{8n} + I_{9n} \end{aligned} \tag{C.4}$$

where

$$I_{1n} \equiv L^{4(n-N)} \sum_p{}'' \frac{\sum_t F_n(p;t)[G_n(p;t)]^2}{\sum_t F_n(p;t)}$$

$$I_{2n} \equiv L^{4(n-N)} \sum_p{}'' \left(\frac{\sum_t F_n(p;t)[G_n(p;t)]}{\sum_t F_n(p;t)}\right)^2$$

$$\sum_p{}'' \equiv \sum_{p \in 2\pi L^{n-N}\mathbf{Z}^4, |p_\nu| < \pi, (\exists \nu, |p_\nu| > \pi/L)}$$

and for $i = 3, 4, ..., 9$,

$$I_{i,n} \equiv L^{4(n+1-N)} \sum_{p \in 2\pi L^{n+1-N}\mathbf{Z}^4, |p_\nu| < \pi} \tilde{I}_{i,n}(p)$$

with

$$\tilde{I}_{3n}(p) \equiv \frac{\sum_{t \neq 0} F_{n+1}(p; Lt)[G_{n+1}(p; Lt)]^2}{\sum_t F_{n+1}(p; Lt)}$$

$$\tilde{I}_{4n}(p) \equiv \frac{\sum_{s \neq 0} F_{n+1}(p; s)[G_{n+1}(p; s)]^2}{\sum_s F_{n+1}(p; s)}$$

$$\tilde{I}_{5n}(p) \equiv \frac{F_{n+1}(p; 0)[G_{n+1}(p; 0)]^2 \sum_{s \notin L\mathbf{Z}^4} F_{n+1}(p; s)}{\sum_s F_{n+1}(p; s) \sum_t F_{n+1}(p; Lt)}$$

$$\tilde{I}_{6n}(p) \equiv \frac{\sum_t F'(Lt)\, G'(Lt)}{\sum_t F'(Lt)} \frac{\sum_{t \neq 0} F'(Lt)\, G'(L(t)}{\sum_t F'(Lt)}$$

$$\tilde{I}_{7n}(p) \equiv \frac{\sum_s F'(s)\, G'(s)}{\sum_s F'(s)} \frac{\sum_{s \neq 0} F'(s)\, G'(s)}{\sum_s F'(s)}$$

$$\tilde{I}_{8n}(p) \equiv \frac{F'(0)\, G'(0) \sum_t \sum_{s,s' \notin L\mathbf{Z}^4} F'(Lt)\, G'(Lt)\, F'(s)\, F'(s')}{\sum_{t,t'} \sum_{s,s'} F'(Lt)\, F'(Lt')\, F'(s)\, F'(s')}$$

$$\tilde{I}_{9n}(p) \equiv \frac{F'(0)\, G'(0) \sum_{t,t'} \sum_{s \notin L\mathbf{Z}^4} F'(s)\, G'(s)\, F'(Lt)\, F'(Lt')}{\sum_{t,t'} \sum_{s,s'} F'(Lt)\, F'(Lt')\, F'(s)\, F'(s')}$$

Now for these I_{in}, we can prove:

Lemma C.2. (i) $0 \leqslant I_{i,n}$ $(i = 1, 2, ..., 9)$.

(ii) $I_{2n} \leqslant I_{1n}$ and $I_{8n}, I_{9n} \leqslant I_{5n}$.

(iii) For $L \geqslant 100$,

$$0.011 \ln 0.095L + 0.007 \leqslant I_{1n} \leqslant \tfrac{1}{6} \ln L + 0.044$$

$$I_{3n} \leqslant (4L-2)^{-4}, \qquad I_{4n} \leqslant 0.004, \qquad I_{5n} \leqslant 6.063$$

$$I_{6n} \leqslant \tfrac{7}{5}(2L-1)^{-2}, \qquad I_{7n} \leqslant 0.164$$

(iv) For $L \geqslant 100$, $N \geqslant n+40$, and $n \geqslant 10$,

$$|I_{i,n}/I_{i,n+1} - 1| \leqslant L^{20-(N-n)/2} + L^{5-n/2}$$

Proof. (i) Obvious ($F_n, G_n \geqslant 0$).

(ii) Compare $\tilde{I}_{1n}$ and $\tilde{I}_{2n}$. Defining

$$\langle \cdots \rangle \equiv \frac{\sum_t F(t) \cdot (\cdots)}{\sum_t F(t)}$$

we have

$$\tilde{I}_{1n}(p) = \langle \{G(t)\}^2 \rangle_F \geqslant (\langle \{G(t)\} \rangle_F)^2 = \tilde{I}_{3n}(p) \qquad \text{(C.5)}$$

The second inequality is proven by explicitly calculating the difference between I_5 and I_8 or I_9, and using

$$G_{n+1}(p; s) \leqslant G_{n+1}(p; 0)$$

(iii) Straightforward calculation. Use, on occasion, representations like (C.5).

(iv) Because all $I_{i,n}$ are well defined and finite and have limits as $N-n \to \infty$ and $n \to \infty$, this is a quite reasonable conclusion. To prove this, use

$$\min_i \frac{A_i}{B_i} \leqslant \frac{\sum_i A_i}{\sum_i B_i} \leqslant \max_i \frac{A_i}{B_i} \qquad \text{for} \quad A_i, B_i > 0$$

to bound the ratio $I_{i,n}/I_{i,n+1}$ by the ratio of integrands $I_{i,n}(p)/I_{i,n+1}(p)$ for $|p_v| \geqslant L^{n-N}$. The contribution from $|p_v| < L^{n-N}$ is extremely small (relatively of order $L^{(n-N)/2}$). ▌

Now, by Lemma C.2, we can easily prove Proposition C.1. By (i) and (ii), (C.4) can be bounded as

$$\begin{aligned} I_n &\leqslant 2(I_{1n} + I_{3n} + I_{5n}) + I_{7n} + I_{9n} \\ &\leqslant 2I_{1n} + 3I_{5n} + 2I_{3n} + I_{7n} \leqslant 37/2 + \tfrac{1}{3} \ln L \end{aligned}$$

Also,

$$I_n \geqslant I_{1n} - 2I_{4n} - I_{6n}$$

As for the limiting property, choose $\underline{n} \geqslant 15$ and $N \geqslant 2\underline{n} + 30$, and consider $15 \leqslant n \leqslant \underline{n}$. Then, writing

$$\frac{I_n}{I_{n+1}} = \frac{1}{I_{n+1}} \left(2I_{1,n+1} \frac{I_{1,n}}{I_{1,n+1}} + \cdots \right)$$

and using the bounds (iv), we have

$$|I_n/I_{n+1} - 1| \leqslant \tfrac{1}{2}(L^{22-(N-n)/2} + L^{7-n/2}) \leqslant L^{7-n/2}$$

Thus, for $15 \leqslant n \leqslant \underline{n}$,

$$\frac{I_n}{I_{\underline{n}}} = \prod_{k=n}^{\underline{n}-1} \frac{I_k}{I_{k+1}} \begin{cases} \leqslant \prod_{k=n}^{\underline{n}-1} (1 + L^{7-k/2}) \\ \geqslant \prod_{k=n}^{\underline{n}-1} (1 - L^{7-k/2}) \end{cases}$$

Taking the logarithm (note that $7 - k/2 \leqslant -1/2$), we obtain

$$|\ln(I_n/I_{\underline{n}})| \leqslant \sum_{k=n}^{\underline{n}-1} [L^{7-k/2} + (L^{7-k/2})^2] \leqslant 2L^{7-n/2} \quad ∎$$

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NOTES ADDED IN PROOF

In this paper, we have introduced (as in Ref. 16) an infrared regulator $\xi = 1$ to make massless Gaussian propagator G_n on a finite torus well-defined. Note that Gaussian kernels $\mathscr{A}_n$, Q_n, $\mathscr{T}_n$ do not depend on ξ. Also note that the *inverse* of G_n is well-defined *without* the infrared regulator.

Rigorously speaking, Eq. (2.30) of Ref. 16 (or Eq. (4.4) of this paper) holds only for G_n^{-1} *without* the massless mode. To use Eq. (4.4), we add and subtract $\xi L^{2n-4(N-n)}(\sum \varphi_x^n)^2$ from the effective Hamiltonian and express its Gaussian part as $(\Phi^n, G_n^{-1}, \Phi^n) = (\Phi^n, G_n^{(;L^{2n}\xi)^{-1}}, \Phi^n) - \xi L^{2n-4(N-n)}(\sum \varphi_x^n)^2$. Now the first term is handled by (4.4), while the second term is exactly transformed into $-\zeta\xi L^{2n+2-4(N-n-1)}(\sum \varphi^{n+1})^2$. If we combine these two, we can obtain Eqs. (3.1) and (4.8) with Gaussian propagators G_n^{-1} *without* the infrared regulator.

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